

COUPLING FUNCTIONS FOR DOMINO TILINGS OF AZTEC DIAMONDS

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ABSTRACT. The inverse Kasteleyn matrix of a bipartite graph holds much information about the perfect matchings of the system such as local statistics which can be used to compute local and global asymptotics. In this paper, we consider three different weightings of domino tilings of the Aztec diamond and show using recurrence relations, we can compute the inverse Kasteleyn matrix. These weights are the one-periodic weighting where the horizontal edges have one weight and the vertical edges have another weight, the q^{vol} weighting which corresponds to multiplying the product of tile weights by q if we add a ‘box’ to the height function and the two-periodic weighting which exhibits a flat region with defects in the center.

1. INTRODUCTION

1.1. Terminology. Domino tilings of bounded lattice regions have been extensively researched during the last twenty years. These tilings are the same as *perfect matchings* of a bounded portion G of the dual square lattice, in the following way: a *matched edge* corresponds to a domino; the fact that the dominos do not overlap means that no two matched edges share a vertex, and the fact that the dominos cover the entire region means that each vertex in the region is covered by a matched edge. In the statistical mechanics literature, one speaks of *dimer covers* rather than *perfect matchings*, and *dimers* rather than *matched edges*.

The most well-studied example of such a model is domino tilings of the *Aztec diamond* which was introduced in [11]. Here, one tiles the region $\{(x, y) : |x| + |y| \leq n + 1\}$ with 2 by 1 rectangles where n is the size of the Aztec diamond. There are other examples of the theory, but they involve replacing the graph G with a different one, such as the regular square-octagon lattice (giving the so-called *diabolo tilings*) or the hexagonal mesh (giving *lozenge tilings*).

By giving each edge a multiplicative weight, we can consider *random* dimer coverings: the probability of each covering is proportional to the product of the edge weights of the dimer covering. The corresponding discrete probability space is called the *dimer model*. If the graph G is bipartite (as it shall be for the rest of this paper) then each dimer covering can be encoded by a three dimensional discrete surface, where the third co-ordinate is derived from the specific dimer covering and is called the *height function* [30]. For random tilings, [9, 21] showed that with probability tending to one, the height function of a randomly tiled large bounded region tends to a deterministic *limit shape*. This shape is not smooth over the entire region: typically, there are macroscopic regions wherein the tiling is “frozen” (i.e.

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exhibits deterministic correlations), called *facets*; as such the measure is often said to be in a *solid state* here [22]. Outside of the facets, the correlations between pairs of dimers are only mesoscopic, tending to zero as the dimers move farther apart. If the decay is polynomial, the measure is said to be *liquid*; if it is exponential it is said to be *gaseous* [22]. Not all tilings possess a gaseous region; however, all but the most degenerate have liquid regions. The limiting height function is smooth in these regions. Figure 1 shows two random tilings of relatively large Aztec diamonds.

1.2. Local asymptotics for nice regions. For particular bounded regions, one approach to study these models uses an interlaced particle system which can be derived from the underlying tiling [15, 3]. Using the Lindström-Gessel-Viennot theorem [16, 29] combined with the Eynard-Mehta theorem (e.g. see [4]), it is often possible to find the correlation kernel for a determinantal process and compute finer statistics for the model [15, 3]. Using these statistics, one can study the fluctuations between the interface of the solid and liquid regions when the system size gets large. Amazingly, these fluctuations have the same distributions arising from the study of eigenvalues of random matrices (see for example [15, 17, 13]). Furthermore, one can even change the boundary conditions of the underlying tiling problem to find more degenerate kernels which also appear in the random matrix literature, for example, see [1].

For bipartite graphs, the *Kasteleyn matrix* is a signed weighted adjacency matrix indexed by the white and black vertices of the graph [18]. The inverse of the Kasteleyn matrix, known as the *inverse Kasteleyn matrix*, for bipartite graphs provides much information about the model – by [19] the edges form a determinantal process with the kernel given by the inverse Kasteleyn matrix. Hence, by knowing the inverse Kasteleyn matrix for a bipartite graph one can compute all finite, local and global asymptotics of the edges in the dimer model. For lozenge tilings, the interlaced particle system kernel can be used to compute the inverse Kasteleyn matrix as the particle system kernel and the inverse Kasteleyn matrix are in bijection [26]. However, for domino tilings on the Aztec diamond, the most natural kernel from the interlaced particle system contains different information to the inverse Kasteleyn matrix does. Although, one can derive the particle system correlation kernel from the inverse Kasteleyn matrix, the particle system correlation kernel gives a better description of the interface between the unfrozen and frozen regions, see [7]. By knowing both the interlaced particle system and the inverse Kasteleyn matrix, we believe that one understands the full asymptotic picture of the system.

1.3. Purpose. The aim of this paper to highlight an elementary procedure which allows one to compute the correlation kernel of the determinantal process associated to the edges of tilings of Aztec diamonds. For pedagogical reasons, we do this first for the most well-understood instance of the dimer model (thereby recovering the work of [14]), followed by two, substantially more complicated new settings: the so-called q^{vol} weighting (Section 4, and the two-periodic weighting 5 which includes as a special case the uniform measure on diabolo tilings on a *fortress graph* [27]. The two-periodic weighting has a 2 by 2 fundamental domain which is defined in [22].

For the q^{vol} weighting, large random tilings of the Aztec diamond possess a limit shape exists when $q \rightarrow 1^-$ as the system size tends to infinity which exists using the results from [21]. However, when $q \rightarrow 1^-$ and the $a \rightarrow \infty$, the results from [21] will no longer apply but simulations seem to suggest that there may be a limit shape and possibly interesting local and global asymptotic behavior. Figure 2 shows relatively large tilings with q^{vol} weighting.

Large random tilings of the two-periodic weighting of the Aztec diamond feature all three phases where the gaseous region described by an ‘octic’ curve. Explicit formulas for this curve are given derived in [27] and [21] using general machinery. Figure 3 shows a relatively large random tiling of a two-periodic weighting of the Aztec diamond. By having an expression for the inverse Kasteleyn matrix for the two-periodic weighted Aztec diamond, it may be possible to study this model on all the phase interfaces. Computing the interlaced particle system kernel for the two-periodic weighting of the Aztec diamond using the Lindström-Gessel-Viennot theorem [16, 29] combined with the Eynard-Mehta theorem (e.g. see [4]) seems somewhat complicated – one can either proceed by inverting either a block LGV matrix or block Toeplitz matrix.

1.4. Explicit inversion of Kasteleyn matrices. We turn now to a discussion of our methods. It is possible to derive the inverse Kasteleyn matrix for domino tilings of the Aztec diamond with weight 1 for horizontal tiles and weight a for vertical tiles (one-periodic weighting). We also show that it is possible to extend the method to the q -analog of domino tilings of the Aztec diamond (q^{vol} weighting). The height function of a domino tiling can be viewed as the surface of a pile of *Levitov blocks* [23, 24]; one can modify the edge weights so that the weight of each tiling is proportional to $q^{\# \{\text{Levitov blocks}\}}$. Finally, we show that it is possible to compute the inverse Kasteleyn matrix with a two-periodic weighting of the Aztec diamond and do so for an Aztec diamond of size $4m$ for $m \in \mathbb{N}$.

It is often quite difficult to invert a matrix whose entries have parameters; indeed, the typical methods in the literature involve first orthogonalizing, so that the matrix to be inverted is diagonal [5]. However, in the particular case of the Aztec diamond, one can make some progress by computing a multivariate generating function for the entries of K^{-1} . This is possible essentially using graphical transformations similar to those used in [11] which is the same procedure used in the generalized domino shuffle [27] to compute the weights in the transitional steps of the shuffling algorithm.

We use three relations among the entries of K^{-1} . The first two are the matrix identities $K.K^{-1} = \mathbb{I}$, $K^{-1}.K = \mathbb{I}$; the third is a recurrence relation in n , the order of the Aztec diamond, generated from the generalized domino shuffling algorithm which determines only those elements of K^{-1} which correspond to two dimers on the *boundary* of the region. We solve this recurrence by computing the ordinary power-series generating function for its coefficients, which we call the *boundary generating function*.

Having done this, the entries of K^{-1} are slightly overdetermined. As such, we treat the equations $K.K^{-1} = \mathbb{I}$, $K^{-1}.K = \mathbb{I}$ as *linear recurrence relations*, for which the *boundary generating function* serves as a boundary condition.

1.5. Details. The heuristic in the preceding sections drastically oversimplifies the intricacy of the combinatorics involved in these calculations, which consume the bulk of this paper. We also found that the computation of the boundary generating function of the two-periodic weighting was intractable without the use of computer algebra. Here is a description of the calculation in somewhat greater detail. The recurrences $K.K^{-1} = K^{-1}.K = I$ allow us to write each entry of the inverse Kasteleyn matrix as a linear combination of entries of $K^{-1}(\mathbf{w}, \mathbf{b})$ where \mathbf{b} and \mathbf{w} are black and white vertices on the boundary. To find $K^{-1}(\mathbf{w}, \mathbf{b})$ where \mathbf{b} and \mathbf{w} are black and white vertices on the boundary, we first find $Z(\mathbf{w}, \mathbf{b})/Z$ where Z is the partition function (the sum of the weights of all tilings) and $Z(\mathbf{w}, \mathbf{b})$ is the partition function of the tiling with the vertices \mathbf{b} and \mathbf{w} removed.

As removing two vertices \mathbf{w} and \mathbf{b} from the boundary does not change the so-called *Kasteleyn orientation*, we are able to recover $K^{-1}(\mathbf{w}, \mathbf{b})$ from $Z(\mathbf{w}, \mathbf{b})/Z$. The boundary recurrence relation gives a relation for $Z(\mathbf{w}, \mathbf{b})/Z$ as a linear combination of $Z(\mathbf{w}_1, \mathbf{b}_1)/Z$ for a smaller Aztec diamond where w, w_1 and b, b_1 are white and black vertices on the boundaries of their corresponding Aztec diamonds. To solve this recurrence, we use a generating function approach and hence find $K^{-1}(\mathbf{w}, \mathbf{b})$ for \mathbf{b} and \mathbf{w} are black and white vertices on the boundary. However, for the q^{vol} weighting, it is simpler to solve the boundary recurrence relation directly using double contour integral which we obtained from its boundary generating function.

For the one-periodic and q^{vol} weighting, we can use the three sets of recurrence relations mentioned above to find a formula for the inverse Kasteleyn matrix. For the one-periodic weighting, we write the formula as a generating function. For the q^{vol} weighting, we obtain a double contour integral. Indeed, sending $q \rightarrow 1$, we find a double contour integral formula of the inverse Kasteleyn matrix in the one-periodic case, see [7] for a comparison.

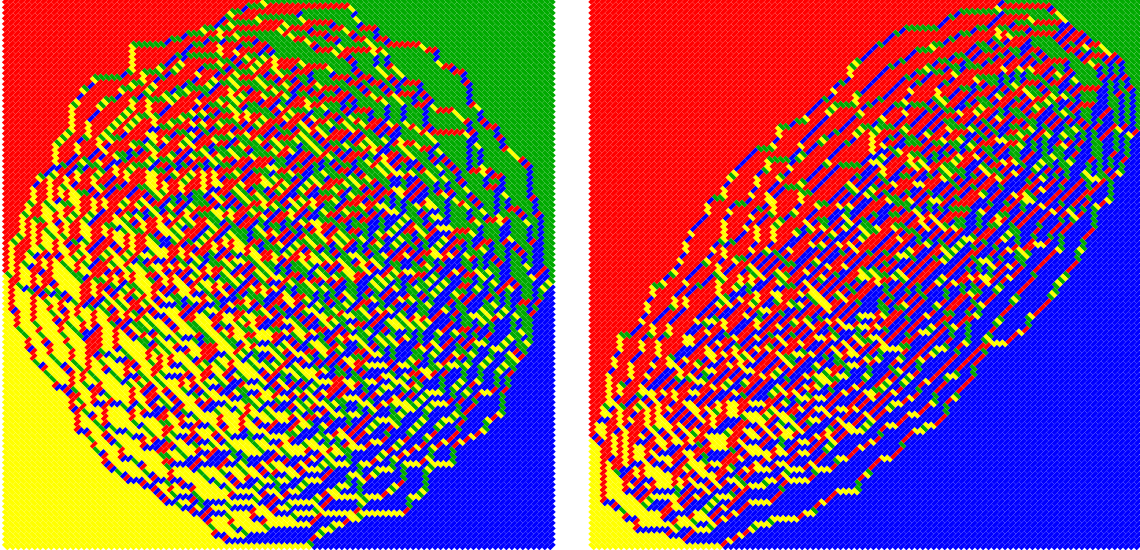
For the two-periodic weighting of the Aztec diamond, we must use a further two recurrence relations; $K^*K.K^{-1} = K^*\mathbb{I}$ and $K^{-1}.K.K^* = \mathbb{I}.K^*$ due to the complexity of the model; of course, these are completely equivalent and in principle carry no new information; however, they are simpler in a certain sense as they involve the *discrete Laplacian*. Additionally, the boundary recurrence relation for the two-periodic weighting of the Aztec diamond has order 4 with the size of the Aztec diamond and is dependent on parity of the white and black vertices on the boundary. This leads to a matrix equation for the recurrence relations describing the boundary generating function. We only find the solution of the matrix recurrence relation when the size of the Aztec diamond is equal to $4m$, though in principle the “only obstacle” to handling the other sizes of diamond is the difficulty of the computations. This leads a formula for the inverse Kasteleyn matrix when the size of the Aztec diamond is equal to $4m$.

1.6. Remarks about Asymptotics. As remarked above, this paper might provide a gateway to computing fine asymptotics of the q^{vol} and two-periodic weightings of the Aztec diamond. That is, one will hopefully be able to compute global correlations and local correlations at the phase boundaries using the formulas found in this paper.

From simulations, the q^{vol} weighting seems to exhibit interesting limiting behavior when $q \rightarrow 1^-$ and also when $q \rightarrow 1^-$ and $a \rightarrow \infty$ as $n \rightarrow \infty$ simultaneously. We believe that a possible parametrization would be to set $a = e^{c/2}$ and $q = e^{-c/n}$ where n tends to infinity. Although it is possible to derive to the limit shape using [21], we think our formula could be used to find the height fluctuations in the unfrozen region when $c > 0$. When $c = 0$, the model is equivalent to the one-periodic weighting of the Aztec diamond and so the height fluctuations are governed by the so-called *Gaussian Free field* (details of this process can be found in [28]) as shown in [7]. When c tends to infinity, we suspect that the unfrozen region is a *flat square* given by alternating (diagonal) columns of east and west dominos. That is, for a rescaled Aztec diamond with corners given by $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$, the unfrozen region is given by $(1/2, 0), (0, 1/2), (1/2, 1)$ and $(1, 1/2)$. From initial computations, the asymptotic analysis to find these height fluctuations is encouraging and is current work in progress.

It may also be possible to derive the q^{vol} correlation (particle) kernel using established techniques (e.g. [15, 3]). This works quite cleanly, since the process in question is *Schur* [25], although, we have not tried this computation.

FIGURE 1. Random Simulations of the Aztec diamond of size 100 with one-periodic weight for $a = 1$ (left) and $a = 1/2$ (right)(see Section 3 for a description of edge weights).



For the two-periodic weighting, the process does not appear to be Schur. As mentioned above, under the right choice of parameters the model exhibits a third phase which has been named gaseous in which the height function correlations decay exponentially [22]. Other models that might possess similar phenomenon are the three periodic lozenge tiling in a large hexagon and the six vertex model with domain wall boundary conditions away from the so-called free fermion point [10]. Indeed, the six vertex model at its free fermion point with domain boundary conditions can be recovered from the one periodic Aztec diamond [12].

The main motivation behind this paper was to find the correlation kernel of the two-periodic weighting of the Aztec diamond, so that one can find the local correlations at the gaseous-liquid boundary. As mentioned above, this boundary represents the transition from the correlations of dominos having power law decay to the correlations of dominos having exponential decay. For tiling models, the solid-liquid boundary (with no cusps) has been well studied (see [26] for the most general results); the interlaced particle system associated with the tiling has fluctuations of size $n^{1/3}$ and the distributions of particles are normally given by the so-called *Airy process*, a natural distribution originally formulated in Random matrix theory [2]. As far as we are aware, the gaseous-liquid boundary of a tiling model has not been previously studied in any such probabilistic model.

The formula for the inverse Kasteleyn matrix for two-periodic weightings of the Aztec diamond, is given as a four variable generating function and it is not immediate how to analyze asymptotically. However, Kurt Johansson, using our formula, has been able to derive a double contour integral formula. From this double contour integral formula, it should be possible to use a saddle point analysis. Indeed, early computations using this approach show that the limiting octic curve can be recovered which agrees with limiting curve computed using the techniques from [21]. We believe that this approach will lead to

FIGURE 2. Random simulations of the Aztec diamond of size 200 with q^{vol} weighting (see Section 4). The top picture has weights $q = 0.99$ and $a = 1$. The picture below has $q = 0.98$ and $a = 10$.

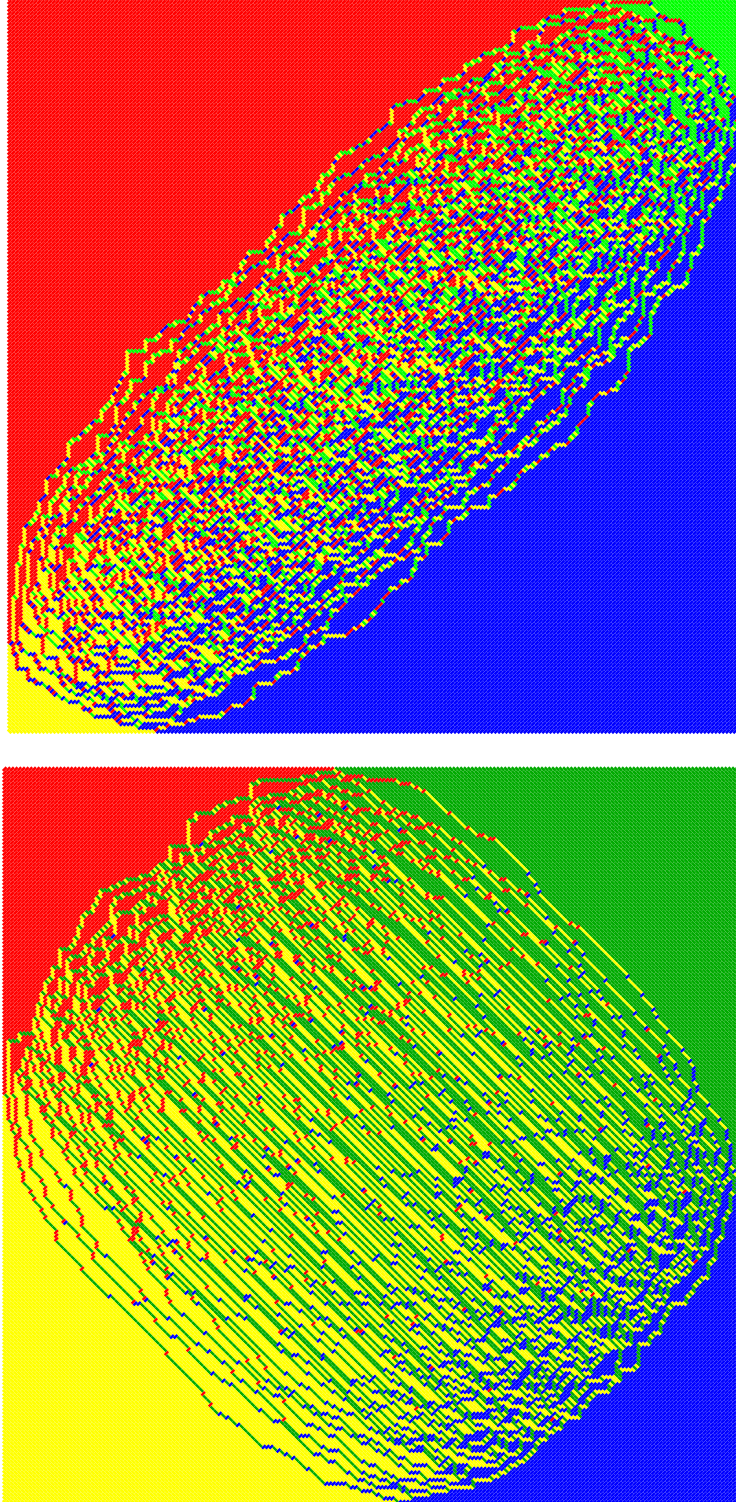


FIGURE 3. Random simulations of the Aztec diamond of size 200 with two-periodic weights (see Section 5. The first picture has $a = 1/2$ and $b = 1$ with 8 colors. The second picture is the same tiling as the first but contains four colors to highlight the three phases for black and white printing.

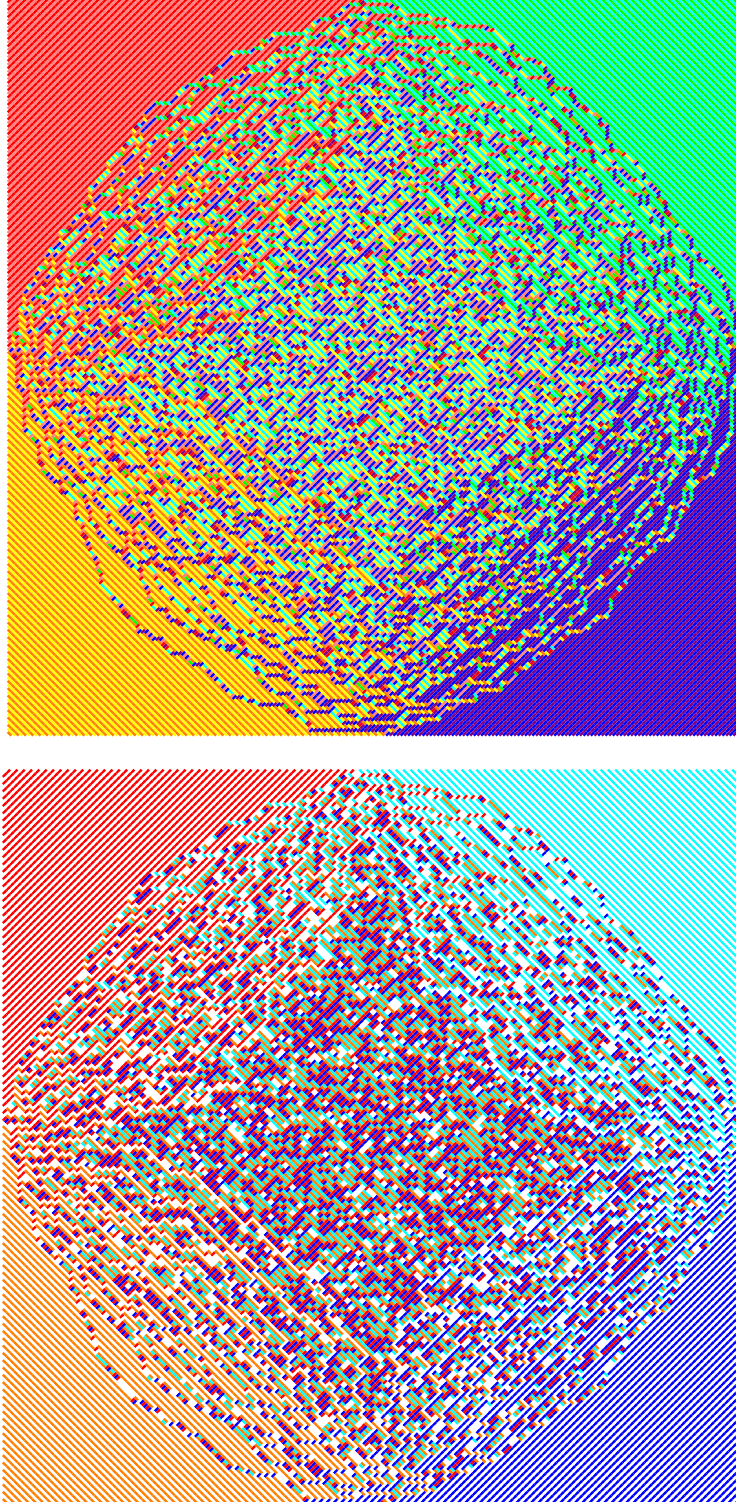
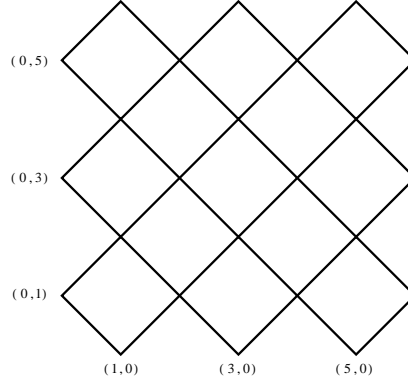


FIGURE 4. The Kasteleyn co-ordinates for an Aztec diamond of order 3.



finding the gaseous-liquid fluctuations. This is current work in progress which will hopefully appear elsewhere [6].

1.7. Overview of the Paper. The paper is organized as follows: Section 2 we give some of the prerequisites and notation for understanding the proofs of the rest of the paper. In Section 3, we compute the generating function of the inverse Kasteleyn matrix for one-periodic Aztec diamonds which provides a blueprint for the computing. In Section 4, we give a contour integral formula for the inverse Kasteleyn matrix for q^{vol} weighting of an Aztec diamond. Finally in Section 5 we derive the generating function for two-periodic weightings of the Aztec diamond for size $4m$.

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2. NOTATION AND BACKGROUND INFORMATION

Let $W = \{(x_1, x_2) : x_1 \bmod 2 = 1, x_2 \bmod 2 = 0, 1 \leq x_1 \leq 2n - 1, 0 \leq x_2 \leq 2n\}$ and let $B = \{(x_1, x_2) : x_1 \bmod 2 = 0, x_2 \bmod 2 = 1, 0 \leq x_1 \leq 2n, 1 \leq x_2 \leq 2n - 1\}$. The set $W \cup B$ denotes the vertex set of the dual graph of the Aztec diamond (rotated by $-\pi/4$ and translated), where W denotes the set of white vertices and B denotes the set of black vertices. We call the above co-ordinate system (of the dual graph) of the Aztec Diamond, the *Kasteleyn co-ordinates* - see Figure 4 for an example. Let $e_1 = (1, 1)$ and $e_2 = (-1, 1)$.

For a planar bipartite graph $G = (V, E)$, the *Kasteleyn orientation* is a choice of edge weights so that the product of edge weights around each face is odd. In this paper, G is the graph formed from (the dual graph of) the Aztec diamond. We consider the edge weights given by positive numbers for edges parallel to e_1 and positive numbers multiplied by $i = \sqrt{-1}$ for edges parallel to e_2 . As G is a bipartite graph, we consider the *Kasteleyn matrix*, whose rows are indexed by black vertices and columns indexed by white vertices,

with entries given by

$$(2.1) \quad K(\mathbf{b}, \mathbf{w}) = \begin{cases} wt(\mathbf{e}) i^{j-1} & \text{for } \mathbf{e} = (\mathbf{b}, \mathbf{w}) \text{ and } \mathbf{b} - \mathbf{w} = e_j, j \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

where $wt(\mathbf{e})$ is edge weight of edge \mathbf{e} . As G is a bipartite finite graph, $|\det K|$ is equal to the number of weighted dimer covers of G . This was first proved by Kasteleyn [18] but in a more general setting. The explicit formula for the inverse Kasteleyn matrix and correlation kernel of the dominos is given as follows: suppose that $e_1 = (b_1, w_1), \dots, e_n = (b_n, w_n)$ then the probability of seeing a perfect matching with the edges e_1, \dots, e_n is given by [19]

$$(2.2) \quad \det (K(b_i, w_i) K^{-1}(w_i, b_j))_{i,j=1}^n.$$

That is, the edges form a determinantal point process with the correlation kernel given by $L(e_i, e_j) = K(b_i, w_i) K^{-1}(w_i, b_j)$ [19]. In the case of the two-periodic weighting of the Aztec diamond, this point process is a block determinantal point process. With an explicit formula for the inverse Kasteleyn matrix, equation (2.2) means that we can compute any joint probability of a subset of edges of the matching which includes the probability of an edge.

We now summarize the graph theory techniques used in this paper.

Two dimer models are said to be *gauge equivalent* if their partition function differs by a constant multiple. The dimer model obtained from multiplying all the edge weights surrounding one specific vertex by the same constant is called a *gauge transformation*. Gauge transformed dimer models are gauge equivalent.

Other than gauge transformations, we make use of three other graph transformations. All three of these alter the graph itself, but leave the partition function invariant up to a gauge transformation.

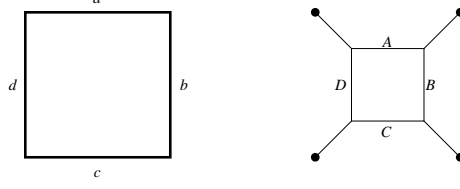
- (1) Suppose we have a large square with edge weights a, b, c and d where the labelling is clockwise, contained in some graph H . Suppose we deform this large square to a smaller square with edge weights A, B, C and D (same labelling convention as the large square) and also include an edge, with edge weight 1, between each vertex of the smaller square and its corresponding original vertex — see Figure 5. We call this new graph H^* . We set $A = c/(ac + bd)$, $B = d/(ac + bd)$, $C = a/(ac + bd)$ and $D = b/(ac + bd)$ so that the local configurations and the weights of the matchings of H are preserved under the transformation to H^* . For example, a dimer covering the edge with weight a and no dimer covering the edge with weight c in H corresponds to dimers covering the edge with weight C and the two diagonal edges incident to the edge with weight A in H^* . This graphical transformation is called *urban renewal* [27] and we have that

$$Z_H = (ac + bd) Z_{H^*}$$

where Z_H and Z_{H^*} are the partition functions of H and H^* respectively.

- (2) If a vertex, v , is incident to two edges, each having weight 1, we can contract the two incident edges and vertices of v , to v . The new edge set of v is the union of the edge set of the two contracted vertices omitting the two edges previously incident to v , all with the same edge weight before the contraction. This procedure is called *edge contraction* and has no effect on the partition function of the dimer covering.
- (3) If a vertex is incident to one edge, i.e. it is a *pendant vertex*, then this vertex, the incident edge and its incident vertex can be removed. This procedure is called removal of *pendant edges*. The partition function of the new graph formed by this

FIGURE 5. The Urban renewal step which maps the large square on the left to the smaller square on the right and multiplies the partition function by $ac+bd$. The diagonal edges have weight 1



procedure is equal to the original partition function divided by the edge weight of the removed edge.

The above three procedures can be used to compute the partition function [11], the edge probabilities [8, 27] and the inverse Kasteleyn matrix for one-periodic, two-periodic and q^{vol} Aztec diamonds. For general weightings of the Aztec diamond and also other stepped square lattices, although it may be theoretically possible to find the inverse Kasteleyn matrix, we were not able to find suitable recurrence relations.

3. ONE PERIODIC CASE

In this section, we derive the inverse Kasteleyn matrix for uniform tilings as it provides a simplest example for our new method. The Inverse Kasteleyn matrix for the one-periodic weighting of the Aztec diamond was originally computed for uniform tilings in [14]. In [7], the formula was generalized to a biased one-periodic weighting using a one-to-two lifting from the interlaced particle system [15]. The new results here are:

- the *boundary generating function* for the Aztec diamonds, defined below, and
- the observation that the generating function for the entries of K^{-1} , for all Aztec diamonds together, is a rational function in six variables: two variables marking the row of K^{-1} , two marking the column of K^{-1} , one marking the size of the Aztec diamond, and the parameter a .

For the setup, it is just as easy to consider the Kasteleyn matrix with horizontal tiles given weight 1, and vertical tiles given weight a (later, we will take $a = 1$ for simplicity). The Kasteleyn matrix for the one-periodic Aztec diamond is given by

$$K(x, y) = \begin{cases} 1 & \text{if } x - y = \pm e_1 \\ ai & \text{if } x - y = \pm e_2 \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbf{B}$ and $y \in \mathbf{W}$.

Let $f_n(x) = (1 - x^n)/(1 - x)$ and let

$$F_n(w, b, a) = -i/(1 + a^2)af_n\left(\frac{(1 + bai)(1 + wai)}{1 + a^2}\right).$$

Further, for $\mathbf{w} = (w_1, w_2)$ and $\mathbf{b} = (b_1, b_2)$ set

$$(3.1) \quad F_n^{0,0}(\mathbf{w}, \mathbf{b}, a) = F_n(w_1^2, b_2^2, a)w_1b_2,$$

$$(3.2) \quad F_n^{0,2n}(\mathbf{w}, \mathbf{b}, a) = F_n(-1/w_1^2, -b_2^2, a)w_1^{2n-1}b_1^{2n}b_2i/a,$$

$$(3.3) \quad F_n^{2n,0}(\mathbf{w}, \mathbf{b}, a) = F_n(-w_1^2, -1/b_2^2, a)w_1w_2^{2n}b_2^{2n-1}i/a$$

and

$$(3.4) \quad F_n^{2n,2n}(\mathbf{w}, \mathbf{b}, a) = F_n(1/w_1^2, 1/b_2^2, a) w_1^{2n-1} w_2^{2n} b_1^{2n} b_2^{2n-1}.$$

Let

$$(3.5) \quad G_n(\mathbf{w}, \mathbf{b}) = \sum_{x \in \mathbb{W}, y \in \mathbb{B}} \mathbf{w}^x \mathbf{b}^y K^{-1}(x, y)$$

where $\mathbf{w}^x = w_1^{x_1} w_2^{x_2}$ and $\mathbf{b}^y = b_1^{y_1} b_2^{y_2}$ denote the generating function of the inverse Kasteleyn matrix of the one-periodic domino tilings of the Aztec diamond with Kasteleyn orientation given by multiplying all vertical edges (the vector e_2 is vertical) by \mathbf{i} .

Theorem 3.1.

$$(3.6) \quad \begin{aligned} G_n(\mathbf{w}, \mathbf{b}) = & \frac{w_1 w_2 b_2 w_2 f_{n+1}(w_1^2 b_1^2) f_n(w_2^2 b_2^2)}{C(w_1, w_2)} \\ & + (1 + \mathbf{i} a w_1) \frac{(1 + \mathbf{i} a b_2^2) F_n^{0,0} + b_1^2 (b_2 w_1 f_n(b_1^2 w_1^2) + (\mathbf{i} a + b_2^2) F_n^{0,2n})}{C(w_1, w_2) C(b_1, b_2)} \\ & + (\mathbf{i} a + w_1^2) w_2^2 \frac{b_1^2 b_2^{2n+1} w_1 w_2^{2n} f_n(b_1^2 w_1^2) + (1 + \mathbf{i} a b_2^2) F_n^{2n,0} + b_1^2 (\mathbf{i} a + b_2^2) F_n^{2n,2n}}{C(w_1, w_2) C(b_1, b_2)} \end{aligned}$$

where $C(r_1, r_2) = 1 + r_1^2 r_2^2 + \mathbf{i} a (r_1^2 + r_2^2)$ and $F_n^{i,j} = F_n^{i,j}(\mathbf{w}, \mathbf{b}, a)$.

The above generating function is a four variable generating function which is dependent on the size of the Aztec diamond. We could include a fifth variable marking the size of the Aztec diamond but that formula is less tractable.

3.1. Boundary Generating function. Our method involves first finding a generating function for only those entries of K^{-1} whose row and column indices correspond to vertices on the boundary of the aztec diamond, called the *Boundary generating function*. This will be a generating function in three variables: two marking two location on the boundary, and a third marking the size of the aztec diamond.

Consider an Aztec diamond with all edge weights equal to 1. Let $Z_n = 2^{n(n+1)/2}$ which is the number of dimer coverings of an Aztec diamond of size n . For white vertices with Kasteleyn co-ordinates given by $(2i+1, 0)$ for $0 \leq i \leq n-1$ and black vertices with Kasteleyn co-ordinates given by $(0, 2j+1)$ for $0 \leq j \leq n-1$, let $Z(i, j, 1, n)$ denote the number of dimer coverings of an Aztec diamond of size n , with edge weights given by 1, with the white vertex $(2i+1, 0)$ and the black vertex $(0, 2j+1)$ removed.

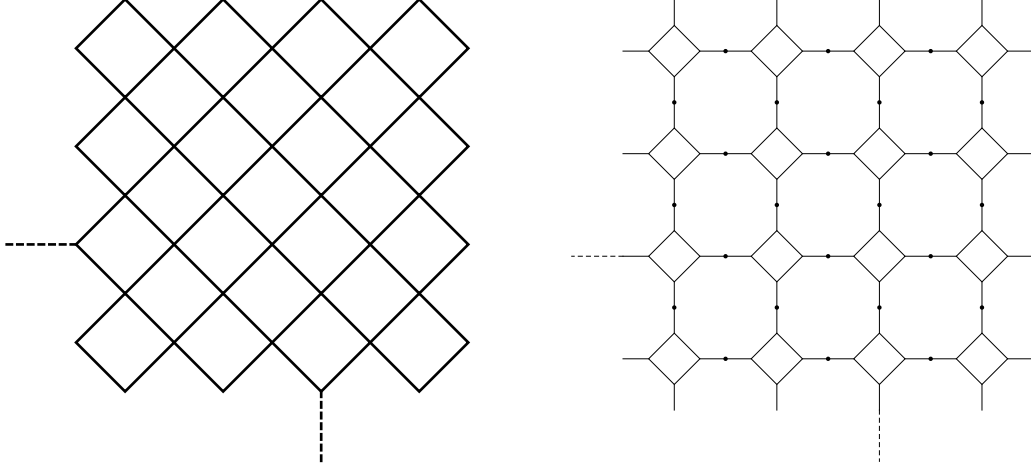
Lemma 3.2.

$$(3.7) \quad \frac{Z(i, j, 1, n)}{Z_n} = \frac{1}{2} \sum_{\substack{k, l \in \{0, 1\} \\ (i-k, j-l) \neq (-1, -1)}} \frac{Z(i-k, j-l, 1, n-1)}{Z_{n-1}} + \mathbb{I}_{(i,j)=(0,0), n \geq 1}$$

Proof. This proof refines the ideas in [27] which compute the edge placement probabilities for the Aztec diamond.

Consider the Aztec diamond with the vertices $(2i+1, 0)$ and $(0, 2j+1)$ removed for $0 \leq i, j \leq n-1$. It is important to notice that the Kasteleyn orientation on the original Aztec diamond, restricted to this graph, remains a Kasteleyn orientation, so the corresponding entry of K^{-1} for the Aztec diamond can be computed from the partition function on this graph; since we have taken $a = 1$ for simplicity, this is just the *number* of matchings.

FIGURE 6. The dashed edges on the left are the two extra edges added as described in the proof of Lemma 3.2 for an Aztec diamond of size 4. The figure on the right is obtained from the figure on the left by applying urban renewal 16 times.



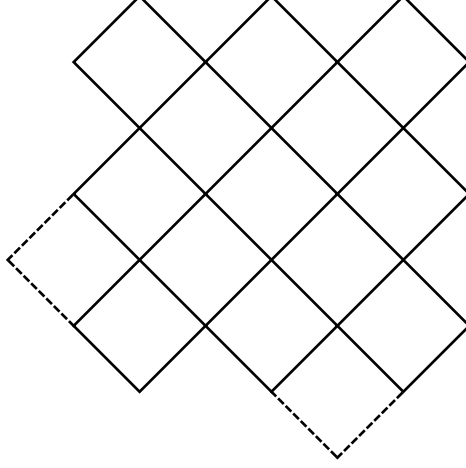
The number of matchings is given by $Z(i, j, 1, n)$ and is in fact equal to the number of matchings of an Aztec diamond with two extra vertices v_0 and v_1 added with an edge between v_0 and $(2i + 1, 0)$ and an edge between v_1 and $(0, 2j + 1)$ for $0 \leq i, j \leq n - 1$. We shall call v_0 and v_1 *auxiliary vertices* and the corresponding edges *auxiliary edges*. To this graph, we apply urban renewal on the faces $(2i + 1, 2j + 1)$ for $0 \leq i, j \leq n - 1$, i.e. we apply urban renewal n^2 times, see Figure 6 for an example. Each edge weight on the urban renewed faces now reads $1/2$ while there is an extra factor of 2 for each urban renewal step. We can remove all the edges connected to vertices incident to one edge due to them always appearing in each matching and apply edge contraction to the vertices incident to two edges. This gives an Aztec diamond of order $n - 1$ with two extra vertices v_2 and v_3 where v_2 has edges to the vertices $(2i + 1, 0)$ and $(2i - 1, 0)$ while v_2 and v_3 has edges to the vertices $(0, 2j + 1)$ and $(0, 2j - 1)$, provided that they are not corner vertices, see Figure 7. These extra edges can be seen from applying the removal of pendant edges to the auxiliary edges. In other words, we have

$$(3.8) \quad Z(i, j, 1, n) = \sum_{k, l \in \{0, 1\}} Z(i - k, j - l, 1/2, n - 1) 2^{n^2}$$

For the corner vertices under this urban renewal, we have $Z(n - 1, j, 1/2, n - 1) = 0$, $Z(-1, j, 1/2, n - 1) = 0$ provided $j \neq -1$, $Z(i, n - 1, 1/2, n - 1) = 0$ and $Z(i, -1, 1/2, n - 1) = 0$ provided $i \neq -1$ due to these shapes being not tileable. We have that $Z(-1, -1, 1/2, n - 1)$ is equal to the number of dimer coverings of an Aztec diamond of order $n - 1$ with edge weights $1/2$ and we have $2^{n^2} Z(-1, -1, 1/2, n - 1) = 2^n Z_{n-1}$ by performing the gauge transformation which multiplies all the edge weights incident to white vertices by 2. Therefore, we have

$$(3.9) \quad Z(i, j, 1, n) = \sum_{\substack{k, l \in \{0, 1\} \\ (i - k, j - l) \neq (-1, -1)}} Z(i - k, j - l, 1/2, n - 1) 2^{n^2} + \mathbb{I}_{(i, j) = (0, 0)} Z_{n-1} 2^n.$$

FIGURE 7. The result of edge contraction from Figure 6. The four dashed edges represent the result of the edge contraction on the two dashed edges in Figure 6.



We can apply the gauge transformation which multiplies all the edge weights incident to white vertices by 2 to $Z(i - k, j - l, 1/2, n - 1)$. As there are $n(n - 1) - 1$ white vertices, we have

$$(3.10) \quad Z(i, j, 1, n) = \sum_{\substack{k, l \in \{0, 1\} \\ (i-k, j-l) \neq (-1, -1)}} Z(i - k, j - l, 1, n - 1) 2^{n-1} + \mathbb{I}_{(i,j)=(0,0)} Z_{n-1} 2^n.$$

Dividing by Z_n and noting that $Z_n = 2^n Z_{n-1}$ gives the lemma. \square

We can now compute the boundary generating function for removed vertices. Let

$$(3.11) \quad Z(w, b, 1, z) = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{Z(i, j, 1, n)}{Z_n} w^i b^j z^n$$

Lemma 3.3.

$$(3.12) \quad Z(w, b, 1, z) = \frac{2z}{(1 - z)(2 - z(1 + b)(1 + w))}$$

Proof. We can multiply (3.7) by $w^i b^j z^n$ and sum from $0 \leq i \leq n - 1$ and $0 \leq j \leq n - 1$ which gives

$$(3.13) \quad Z(w, b, 1, z) = \frac{1}{2}(1 + b)(1 + w)zZ(w, b, 1, z) + \frac{z}{1 - z}$$

Rearranging the above equation gives the result. \square

We can now use the recurrence relations to find the generating function for the inverse Kasteleyn matrix.

Proof of Theorem 3.1 for $a = 1$. The equations $K.K^{-1} = \mathbb{I}$ and $K^{-1}.K = \mathbb{I}$ give two recurrence relations. We have

$$(3.14) \quad K^{-1}(x+e_1, y)\delta_{x_1 < 2n} + \mathbf{i}K^{-1}(x+e_2, y)\delta_{x_1 > 0} + \mathbf{i}K^{-1}(x-e_2, y)\delta_{x_1 < 2n} + K^{-1}(x-e_1, y)\delta_{x_1 > 0} = \delta_{x=y}$$

where $x, y \in \mathbf{B}$, $x = (x_1, x_2), y = (y_1, y_2)$ and

$$(3.15) \quad \delta_{x_1 > 0} = \begin{cases} 1 & \text{if } x_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

We also have for $x, y \in \mathbf{W}$

$$(3.16) \quad K^{-1}(x, y+e_1)\delta_{y_1 < 2n} + \mathbf{i}K^{-1}(x, y+e_2)\delta_{y_2 < 2n} + K^{-1}(x, y-e_1)\delta_{y_2 > 0} + \mathbf{i}K^{-1}(x, y-e_2)\delta_{y_2 > 0} = \delta_{x=y}.$$

We can multiply (3.14) by $w^x b^y$ for $x, y \in \mathbf{B}$ and sum both quantities over \mathbf{B} . Each term on the left-hand side of (3.14) can then be simplified to term involving $G_n(\mathbf{w}, \mathbf{b})$ and a remainder term. For example, the first term of the left-hand side of (3.14) gives

$$(3.17) \quad \begin{aligned} \sum_{x, y \in \mathbf{B}} K^{-1}(x+e_1, y)\delta_{x_1 < 2n} w^x b^y &= \frac{1}{w_1 w_2} \sum_{x \in \mathbf{W}, x_2 \neq 0, y \in \mathbf{B}} K^{-1}(x, y) w^x b^y \\ &= \frac{G_n(\mathbf{w}, \mathbf{b}) - \sum_{\substack{1 \leq x_1 \leq 2n \\ (x_1, 0) \in \mathbf{W}, y \in \mathbf{B}}} K^{-1}((x_1, 0), y) w_1^{x_1} b^y}{w_1 w_2} \end{aligned}$$

Continuing the same procedure on the rest of the terms in (3.14) we obtain

$$(3.18) \quad \begin{aligned} &\left(\frac{1}{w_1 w_2} + \frac{w_1}{w_2} \mathbf{i} + \frac{w_2}{w_1} \mathbf{i} + w_1 w_2 \right) G_n(\mathbf{w}, \mathbf{b}) - \left(\frac{1}{w_1 w_2} + \frac{w_1}{w_2} \mathbf{i} \right) G_n^0(w_1, b_1, b_2) \\ &- \left(w_1 w_2 + \frac{w_2}{w_1} \mathbf{i} \right) G_n^n(w_1, b_1, b_2) w_2^{2n} = \left(\frac{1 - (w_1 b_1)^{2n+2}}{1 - w_1^2 b_1^2} \right) \left(\frac{w_2 b_2 (1 - (w_2 b_2)^{2n})}{1 - w_2^2 b_2^2} \right) \end{aligned}$$

where $G_n^0(w_1, b_1, b_2)$ and $G_n^n(w_1, b_1, b_2)$ are the coefficients of w_2^0 and w_2^{2n} respectively in $G_n(\mathbf{w}, \mathbf{b})$.

It remains to compute $G_n^0(w_1, b_1, b_2)$ and $G_n^n(w_1, b_1, b_2)$. Let

$$(3.19) \quad H_n^{i,j}(\mathbf{w}, \mathbf{b}) = \sum_{\substack{1 \leq x_1 \leq 2n-1, \\ 1 \leq y_2 \leq 2n-1}} K^{-1}((x_1, i), (j, y_2)) w_1^{x_1} w_2^i b_1^j b_2^{y_2}$$

where $i, j \in \{0, 2n\}$. We can compute $G_n^0(w_1, b_1, b_2)$ using the recurrence given in (3.16) and the method described above. We obtain

$$(3.20) \quad \begin{aligned} &\left(\frac{1}{b_1 b_2} + \frac{b_1}{b_2} \mathbf{i} + b_1 b_2 + \frac{b_2}{b_1} \mathbf{i} \right) G_n^0(w_1, b_1, b_2) - \left(\frac{1}{b_1 b_2} + \frac{b_2}{b_1} \mathbf{i} \right) H_n^{0,0}(\mathbf{w}, \mathbf{b}) \\ &- \left(\frac{b_1}{b_2} \mathbf{i} + b_1 b_2 \right) H_n^{0,2n}(\mathbf{w}, \mathbf{b}) = \frac{w_1 b_1 (1 - (w_1 b_1)^{2n})}{1 - (w_1 b_1)^2} \end{aligned}$$

For $G_n^n(w_1, b_1, b_2)$, using (3.16) we obtain

$$(3.21) \quad \begin{aligned} &\left(\frac{1}{b_1 b_2} + \frac{b_1}{b_2} \mathbf{i} + b_1 b_2 + \frac{b_2}{b_1} \mathbf{i} \right) G_n^n(w_1, b_1, b_2) - \left(\frac{1}{b_1 b_2} + \frac{b_2}{b_1} \mathbf{i} \right) H_n^{2n,0}(\mathbf{w}, \mathbf{b}) \\ &- \left(\frac{b_1}{b_2} \mathbf{i} + b_1 b_2 \right) H_n^{2n,2n}(\mathbf{w}, \mathbf{b}) = w_2^{2n} b_2^{2n} \frac{w_1 b_1 (1 - w_1^{2n} b_1^{2n})}{1 - w_1^2 b_1^2} \end{aligned}$$

We can compute $H_n^{i,j}(\mathbf{w}, \mathbf{b})$ using Lemma 3.3 in the following way: let

$$(3.22) \quad H_n(w, b) = -\frac{i}{2} \frac{1 - \left(\frac{(1+wi)(1+bi)}{2} \right)^n}{1 - \frac{(1+wi)(1+bi)}{2}}.$$

We have defined $H_n(w, b)$ to be the boundary generating function of the uniform Aztec diamond with the Kasteleyn orientation with the same co-ordinate labeling given to $Z(w, b, 1, z)$. This function is obtained by computing the z^n coefficient of $Z(w, b, 1, z)$ (a polynomial in w and b) and finding the appropriate sign by evaluating one of the terms.

From $H_n(w, b)$ we can find $H_n^{i,j}$ for $i, j \in \{0, 2n\}$. We find that

$$(3.23) \quad H_n^{0,0}(\mathbf{w}, \mathbf{b}) = H_n(w_1^2, b_2^2) w_1 b_2,$$

$$(3.24) \quad H_n^{0,2n}(\mathbf{w}, \mathbf{b}) = H_n(-1/w_1^2, -b_2^2) w_1^{2n-1} b_1^{2n} b_2 i,$$

$$(3.25) \quad H_n^{2n,0}(\mathbf{w}, \mathbf{b}) = H_n(-w_1^2, -1/b_2^2) w_1 w_2^{2n} b_2^{2n-1} i$$

and

$$(3.26) \quad H_n^{2n,2n}(\mathbf{w}, \mathbf{b}) = H_n(1/w_1^2, 1/b_2^2) w_1^{2n-1} w_2^{2n} b_1^{2n} b_2^{2n-1}.$$

We can substitute (3.20) and (3.21) back into (3.18) and use the expressions obtained from equations (3.23) (3.24) (3.25) and (3.26) to find $G_n(\mathbf{w}, \mathbf{b})$. □

Proof of Theorem 3.1 for all a (sketch). Generalizing this proof to arbitrary a requires no new ideas, but rather increases the complexity and amount of bookkeeping required. As such we have chosen to omit the details, but describe the main changes.

For biased tilings, the boundary generating function and the recurrence relations from $K.K^{-1} = K^{-1}.K = \mathbb{I}$ are modified. We also have that $Z_n = (1 + a^2)^{n(n+1)/2}$.

For the boundary recurrence relation, we can find the recurrence for the South West corner of the diamond, that is, removing the vertices $(2i+1, 0)$ and $(0, 2j+1)$ for $0 \leq i, j \leq n-1$. We find that

$$(3.27) \quad \frac{Z(i, j, a, n)}{Z_n} = \sum_{k, l \in \{0,1\}, (i-k, j-l) \neq (-1, -1)} a^{k+l} \frac{Z(i-k, j-l, a, n-1)}{Z_{n-1}(1+a^2)} + \mathbb{I}_{(i,j)=(0,0)}.$$

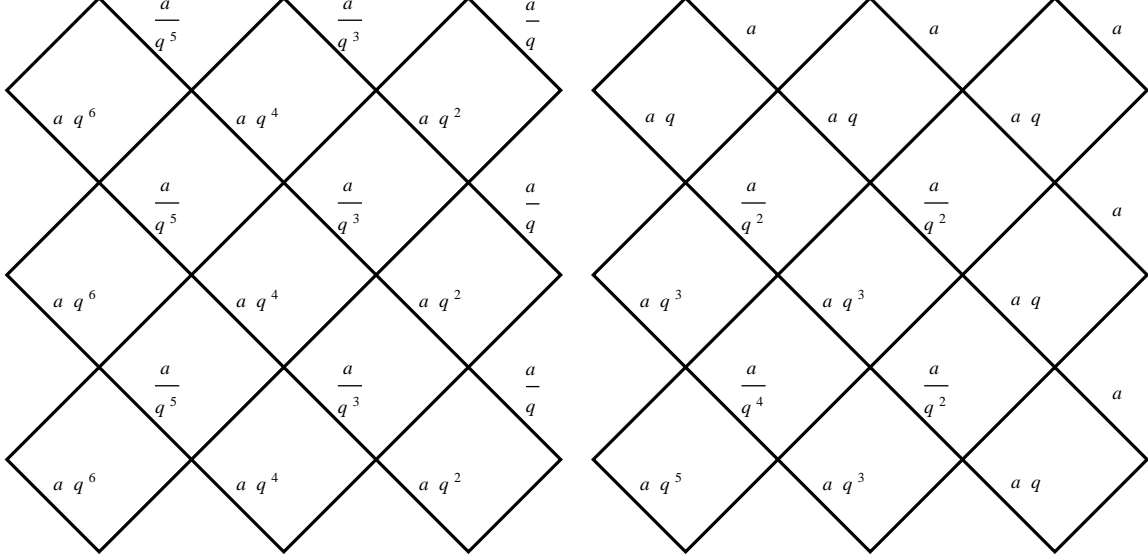
All boundary generating functions for K^{-1} can be computed using manipulations of (3.27) and these are given in equations (3.1), (3.2), (3.3) and (3.4).

The recurrence relations from $K.K^{-1} = \mathbb{I}$ and $K^{-1}.K = \mathbb{I}$ can be used in the same way as described above. □

4. q^{vol} WEIGHTING

As mentioned in the introduction, it is possible to associate a discrete stepped surface to a tiling of the Aztec diamond. This function is called the *height function*. The most concrete way of viewing the height function is as the surface of a certain stack of blocks, called *Levitov blocks* [23, 24]. One can construct an edge weighting of the Aztec diamond in several ways, so that the contribution from each tiling is proportional to $q^{\#\{\text{Levitov Blocks}\}}$ - that is, adding a block to a tiling multiplies the weight of the tiling by q . Concretely, this means that for each face, the product of the edges parallel to e_2 divided by the product of edges parallel to e_1 is equal to q or q^{-1} , depending on the parity of the face. Such a choice of weightings is

FIGURE 8. The two q^{vol} weightings used in this paper for Aztec diamond of size 3. The figure on the left is q^{col} weighting and the figure on the right is the q^{diag} .



not unique, and we refer to any such choice as a q^{vol} *weighting* of the Aztec diamond. For technical reasons, we make use of two such weightings here.

We call the weighting q^{col} to be the choice of weights where all edges parallel to e_1 have weight 1 and the edges parallel to e_2 have edge weights organized in columns given by $aq^{2n}, aq^{1-2n}, aq^{2n-2}, \dots, aq^{-1}$ reading from left to right – see Figure 8. We let K_{col} denote the Kasteleyn matrix with q^{col} weights and its entries are given by

$$(4.1) \quad K_{\text{col}}(x, y) = \begin{cases} 1 & y - x = \pm e_1 \\ aq^{-2n+x_1-1} \mathbf{i} & y - x = e_2 \\ aq^{2n-x_1} \mathbf{i} & y - x = -e_2 \\ 0 & \text{otherwise} \end{cases}$$

where $x = (x_1, x_2) \in \mathbf{B}$ and $y \in \mathbf{W}$.

Theorem 4.1. *The entries of the inverse of K_{col} are given by*

$$(4.2) \quad K_{\text{col}}^{-1}(x, y) = \begin{cases} f_1(x, y) & \text{if } x_1 < y_1 + 1 \\ f_1(x, y) + f_2(x, y) & x_1 \geq y_1 + 1 \end{cases}$$

for $x = (x_1, x_2) \in \mathbf{W}$ and $y = (y_1, y_2) \in \mathbf{B}$, where

$$(4.3) \quad f_1(x, y) = \frac{\mathbf{i}^{(-x_1+x_2+y_1-y_2+2)/2} q^{(4+4n-x_1+x_2-y_1-y_2)(x_1-x_2-y_1+y_2)/4}}{(2\pi \mathbf{i})^2} \int_{\Gamma_{1/a,q}^n} \int_{\Gamma_0} dz dw$$

$$\frac{w^{y_1/2}}{z^{(x_1+1)/2}(w-z)} \frac{\prod_{k=0}^{x_2/2-1} (z + aq^{-2k-1+y_2-1}) \prod_{k=0}^{n-x_2/2-1} (azq^{2k+x_2+3-y_2} - 1)}{\prod_{k=0}^{(y_2-1)/2} (w + aq^{2k-1}) \prod_{k=0}^{n-(y_2+1)/2} (awq^{2k+2} - 1)}$$

and

$$(4.4) \quad f_2(x, y) = \frac{\prod_{k=0}^{(x_1-y_1-3)/2} (-i q^{-2n+x_1-4-2k}) \prod_{k=0}^{(x_2-y_2-3)/2} i q^{2n-x_1+4+2k}}{2\pi i} \int_{\Gamma_{1/a,q}^n} dw$$

$$w^{y_1/2-(x_1+1)/2} \frac{\prod_{k=0}^{x_2/2-1} (w + a q^{-2k-1+y_2-1}) \prod_{k=0}^{n-x_2/2-1} (a w q^{2k+x_2+3-y_2} - 1)}{\prod_{k=0}^{(y_2-1)/2} (w + a q^{2k-1}) \prod_{k=0}^{n-(y_2+1)/2} (a w q^{2k+2} - 1)}$$

where Γ_0 is a contour surrounding the origin and $\Gamma_{1/a,q}^n$ is a contour surrounding $1/(aq^2)$, $1/(aq^4), \dots, 1/(aq^{2n})$ which does not intersect with Γ_0 .

When $q = 1$, we recover the contour integral for domino tilings of the Aztec diamond with edge weights 1 and a for the horizontal and vertical tiles.

4.1. Inverse Kasteleyn Entries on the North and East boundaries. We focus on computing the entries of the inverse Kasteleyn matrix for q^{col} weights on the north boundary (the white vertices $(2k-1, 2n)$ for $1 \leq k \leq n$) and the east boundary (the black vertices $(2n, 2k-1)$ for $1 \leq k \leq n$). For this subsection, label the white vertices on the north boundary 0 to $n-1$ with the label i representing the vertex $(2n-2i-1, 2n)$. Similarly, label the black vertices on the east boundary 0 to $n-1$ with the label j representing the vertex $(2n, 2n-1-j)$.

Although we can find a boundary recurrence relation the q^{col} weighting, it seems intractable to find an explicit solution. This inconvenience can be bypassed by working another (gauge equivalent) choice of q^{vol} weighting.

We let q^{diag} be the q^{vol} weighting where all edges parallel to e_1 have weight 1 and all edges parallel to e_2 on the North and East boundary have weight a which determines the rest of the edge weights – see Figure 8. Let K_{diag} denote the Kasteleyn matrix for the q^{diag} weighting. Its entries are given by

$$(4.5) \quad K_{\text{diag}}(x, y) = \begin{cases} 1 & y - x = \pm e_1 \\ i a q^{-2 \min(n-x_1/2, n-(x_2+1)/2)} & y - x = e_2 \\ i a q^{2 \min(n-x_1/2-1, n-(x_2+1)/2)+1} & y - x = -e_2 \\ 0 & \text{otherwise} \end{cases}$$

where $x = (x_1, x_2) \in B$ and $y \in W$.

Lemma 4.2. *The gauge transformation from q^{col} weighting to q^{diag} weighting is given by multiplying the weight of each edge incident to the white vertex i on the north boundary by $q^{(i+1)^2}$ and the weight of each edge incident to the black vertex j on the east boundary by q^j .*

Proof. Consider the q^{col} weighting with the above multiplication of the edge weights. Fill in the remaining edge weights so that the horizontal edges have weight 1. The weighting recovered is exactly the q^{col} weighting of the Aztec diamond. \square

Let $Z_n^{\text{diag}}(a, q) = Z_n^{\text{diag}}(1; a, a q, a q^{-2}, \dots, a q^{2n-1})$ denote the partition function of tiling the Aztec diamond with q^{diag} weights where 1 is the horizontal edge and $a, a q, a q^{-2}, \dots, a q^{2n-1}$ represent the vertical edges weights on the main diagonal (i.e. edges that intersect the line $(0, 0)$ to $(2n, 2n)$). Let $Z^{\text{diag}}(i, j, a, q, n)$ denote the partition function of tiling the Aztec diamond with q^{diag} weights with vertices $(2n-2i-1, 2n)$ and $(2n, 2n-1-j)$ removed.

Lemma 4.3.

(4.6)

$$\frac{Z_n^{\text{diag}}(i, j, a, q, n)}{Z_n^{\text{diag}}(a, q)} = q^{i+j+1} \left(\sum_{\substack{(k,l) \in \{0,1\} \\ (i-k, j-l) \neq (-1, -1)}} a^{k+l} \frac{Z_n^{\text{diag}}(i-k, j-l, aq, q, n-1)}{(1+a^2q)Z_{n-1}^{\text{diag}}(aq, q)} + \frac{a\mathbb{I}_{(i,j)=(0,0), n \geq 1}}{1+a^2q} \right)$$

Proof. We first find the recurrence relation for the partition function. By using urban renewal, gauge transformation and removing the pendant edges, we find

$$(4.7) \quad Z_n^{\text{diag}}(a, q) = (1+a^2q)^n Z_{n-1}^{\text{diag}}(1; a, aq^3, aq^{-2}, aq^5, \dots, aq^{-2n+4}, aq^{2n-1})$$

Therefore, we have

$$(4.8) \quad \left. \frac{Z_n^{\text{diag}}(a, q)}{(1+a^2q)^n} \right|_{a \rightarrow aq^{-1}} = Z_{n-1}^{\text{diag}}(1; aq^{-1}, aq^2, \dots, aq^{2n-2}).$$

By ensuring that the local moves give the same change to the partition function and that the minimal configuration is the same, we find that

$$(4.9) \quad Z_n^{\text{diag}}(a, q) = (1+a^2q)^n Z_{n-1}^{\text{diag}}(aq, q).$$

The rest of the proof follows from using the same procedures found in Lemma 3.2: we add an auxiliary edge between an added auxiliary vertex and the vertex at $(2n-2i-1, 2n)$ and another auxiliary edge between another added auxiliary vertex and the vertex at $(2n, 2n-2j-1)$. For this new graph, we can use urban renewal, gauge transformation and edge contraction. The factor a^{k+l} given in the statement of the lemma is due to the reduction from $i \mapsto i-1$ (and similarly $j \mapsto j-1$) when $n \mapsto n-1$ removes a vertical edge which has weight a . Similar to computing the partition function recurrence, we have to apply the weight change transformation given in (4.7) and (4.8) for the recurrence of $Z_n^{\text{diag}}(i, j, a, q, n)$ but this changes the weight of the minimal configurations due to the removed vertices $i-k$ and $j-l$ for $k, l \in \{0, 1\}$. To ensure that the minimal configurations remain the same under the reduction, we need to multiply by q^{i+j+1} . \square

Let $Z_n^{\text{col}}(a, q)$ denote the partition function of tiling the Aztec diamond with q^{col} and $Z_n^{\text{col}}(i, j, a, q, n)$ denote the partition function of tiling the Aztec diamond with q^{col} weights with vertices $(2n-2i-1, 2n)$ and $(2n, 2n-1-j)$ removed.

From Lemma 4.3, we can compute a three variable generating function for $Z_n^{\text{diag}}(i, j, a, q, n)/Z_n^{\text{diag}}$ and extract out the relevant coefficient for the size of the Aztec diamond. From this, we can extract out $Z_n^{\text{diag}}(i, j, a, q, n)/Z_n^{\text{diag}}$ using residue calculus. Using a rather non-elementary change of variables, we can find a double integral formula for $Z_n^{\text{diag}}(i, j, a, q, n)/Z_n^{\text{diag}}$. In the next lemma, we show that the double integral formula we obtained solves the recurrence and so we obtain a formula for $Z_n^{\text{col}}(i, j, a, q, n)/Z_n^{\text{col}}$.

Lemma 4.4.

$$(4.10) \quad \frac{Z^{\text{col}}(i, j, a, q, n)}{Z_n^{\text{col}}} = \frac{q^{(2+i+j)^2-1}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{a,q}^n} dz dw \frac{w^n \prod_{k=0}^{n-1} aq^{-2k-1+2(n-j-1)} + z}{(w-z)z^{n-i} \prod_{k=0}^{n-j-1} (aq^{2k-1} + w) \prod_{k=0}^j (-1 + awq^{2k+2})}$$

Proof. The gauge transformation from the proof of Lemma 4.2 gives

$$(4.11) \quad q^{(i+1)^2+j} \frac{Z^{\text{diag}}(i, j, a, q, n)}{Z_n^{\text{diag}}} = \frac{Z^{\text{col}}(i, j, a, q, n)}{Z_n^{\text{col}}}.$$

Let

$$(4.12) \quad f(i, j, a, q, n) = \frac{q^{(1+j)(2+2i+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{a,q}^n} dw dz \frac{w^n \prod_{k=0}^{n-1} (aq^{-2k-1+2(n-j-1)} + z)}{(w-z)z^{n-i} \prod_{k=0}^{n-j-1} (aq^{2k-1} + w) \prod_{k=0}^j (-1 + awq^{2k+2})}$$

We need to show that

Claim 1.

$$(4.13) \quad \frac{Z^{\text{diag}}(i, j, a, q, n)}{Z_n^{\text{diag}}} = f(i, j, a, q, n)$$

Proof of Claim. We show that $f(i, j, a, q, n)$ satisfies the recurrence given in Lemma 4.3. When $i = j = 0$, we have that the integral with respect to w has one singularity at $1/(aq^2)$. We obtain

$$(4.14) \quad \begin{aligned} f(0, 0, a, q, n) &= \frac{q^2}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{a,q}^n} \frac{w^n \prod_{k=0}^{n-1} (aq^{2k-1} + z)}{z^n (w-z) \prod_{k=0}^{n-1} (aq^{2k-1} + w) (waq^2 - 1)} dw dz \\ &= \frac{q^2}{2\pi i} \int_{\Gamma_0} \frac{\prod_{k=0}^{n-1} (aq^{2k-1} + z)}{z^n (a^{-1}q^{-2} - z) \prod_{k=0}^{n-1} (1 + a^2q^{2k+1})} dz \\ &= \frac{q^2}{2\pi i} \int_{\Gamma_0} \frac{\prod_{k=1}^{n-1} (aq^{2k-1} + z)(z + aq^{-1})}{z^n (a^{-1}q^{-2} - z) \prod_{k=0}^{n-1} (1 + a^2q^{2k+1})} dz \\ &= \frac{q^2}{2\pi i} \int_{\Gamma_0} \frac{\prod_{k=1}^{n-1} (aq^{2k-1} + z)}{z^{n-1} (a^{-1}q^{-2} - z) \prod_{k=0}^{n-1} (1 + a^2q^{2k+1})} dz + \frac{aq}{1 + a^2q} \end{aligned}$$

where the fourth equation follows from pushing the integral with respect to z through infinity. By taking the change of variables $w \mapsto q^{-1}w$ and $z \mapsto q^{-1}z$ we can compute $f(0, 0, aq, q, n-1)$. We obtain

$$(4.15) \quad \begin{aligned} f(0, 0, aq, q, n-1) &= \frac{q}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{a,q}^n} \frac{w^{n-1} \prod_{k=1}^{n-1} (aq^{2k-1} + z)}{z^{n-1} (w-z) \prod_{k=1}^{n-1} (aq^{2k-1} + w) (waq^2 - 1)} dw dz \\ &= \frac{q}{2\pi i} \int_{\Gamma_0} \frac{\prod_{k=0}^{n-1} (aq^{2k-1} + z)}{z^{n-1} (a^{-1}q^{-2} - z) \prod_{k=1}^{n-1} (1 + a^2q^{2k+1})} dz \end{aligned}$$

We can compare equations (4.14) and (4.15). We obtain

$$(4.16) \quad f(0, 0, a, q, n) - \frac{f(0, 0, aq, q, n-1)}{1+a^2q} = \frac{aq}{1+a^2q}.$$

When $i, j \neq 0$, we can write

$$(4.17) \quad f(i, j, aq, q, n) = \frac{q^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} dw dz$$

$$\frac{w^n \prod_{k=0}^{n-1} aq^{-2k+2(n-j-1)} + z}{z^{n-i}(w-z) \prod_{k=0}^{n-j-1} aq^{2k} + w \prod_{k=0}^j aq^{2k+3}w-1}$$

By manipulating the integrand of the above expression, we can write

$$(4.18) \quad f(i, j, aq, q, n-1) + f(i-1, j, aq, q, n-1) = \frac{q^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} dw dz$$

$$\frac{w^{n-1} \prod_{k=0}^{n-2} aq^{-2k+2(n-j-2)} + z}{z^{n-i-1}(w-z) \prod_{k=0}^{n-j-2} aq^{2k} + w \prod_{k=0}^j aq^{2k+3}w-1} \left(1 + aq^{-2(1+j)}z^{-1}\right)$$

$$= \frac{q^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} \frac{w^{n-1} \prod_{k=0}^{n-1} aq^{-2k+2(n-j-2)} + z}{z^{n-i}(w-z) \prod_{k=0}^{n-j-2} aq^{2k} + w \prod_{k=0}^j aq^{2k+3}w-1} dw dz.$$

We can also write

$$(4.19) \quad f(i, j, aq, q, n-1) + f(i, j-1, aq, q, n-1) = \frac{q^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} dw dz$$

$$\frac{w^{n-1} \prod_{k=0}^{n-2} aq^{-2k+2(n-j-1)} + z}{z^{n-i-1}(w-z) \prod_{k=0}^{n-j-1} aq^{2k} + w \prod_{k=0}^{j-1} aq^{2k+3}w-1} \left(aq^{-2(1+i+j)} + \frac{aq^{-2(1+i+2j)}}{z}\right)$$

$$= \frac{q^{(2+2i+j)(1+j)-2i-2j-2}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} \frac{w^{n-1} \prod_{k=0}^{n-1} aq^{-2k+2(n-j-1)} + z}{z^{n-i}(w-z) \prod_{k=0}^{n-j-1} aq^{2k} + w \prod_{k=0}^{j-1} aq^{2k+3}w-1} dw dz.$$

For the above equation, we can make the change of variables $w \mapsto q^2w$ and $z \mapsto q^2z$. We obtain

$$(4.20) \quad f(i, j, aq, q, n-1) + f(i, j-1, aq, q, n-1) = \frac{aq^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} dw dz$$

$$\frac{w^{n-1} \prod_{k=0}^{n-1} aq^{-2k+2(n-j-2)} + z}{z^{n-i}(w-z) \prod_{k=-1}^{n-j-2} aq^{2k} + w \prod_{k=1}^j aq^{2k+3}w-1}.$$

We can add (4.18) and (4.20). We obtain

$$(4.21) \quad f(i, j, aq, q, n-1) + f(i-1, j, aq, q, n-1) + f(i, j, aq, q, n-1) + f(i, j-1, aq, q, n-1)$$

$$= \frac{aq^{(2+2i+j)(1+j)}}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} \frac{w^{n-1} \left(\prod_{k=0}^{n-1} aq^{-2k+2(n-j-2)} + z\right) \left(1 + \frac{aq^{-2(-1+aq^3w)}}{aq^{-2}+w}\right)}{z^{n-i}(w-z) \prod_{k=0}^{n-j-2} aq^{2k} + w \prod_{k=0}^j aq^{2k+3}w-1} dw dz$$

$$= \frac{aq^{(2+2i+j)(1+j)}(1+a^2q)}{(2\pi i)^2} \int_{\Gamma_0} \int_{\Gamma_{aq,q}^{n-1}} \frac{w^n \left(\prod_{k=0}^{n-1} aq^{-2k+2(n-j-2)} + z\right)}{z^{n-i}(w-z) \prod_{k=-1}^{n-j-2} aq^{2k} + w \prod_{k=0}^j aq^{2k+3}w-1} dw dz$$

Finally, we can take the change of variables $w \mapsto q^{-1}w$ and $z \mapsto q^{-1}z$ and multiply by q^{i+j+1} which gives $f(i, j, a, q, n)$ as required. \square

Lemma 4.4 now follows directly. \square

Proof of Theorem 4.1. Using Lemma 4.4, we can write K^{-1} for the entries on the NW boundary using the Kasteleyn co-ordinates and adding the relevant sign from the Kasteleyn orientation. We obtain

$$(4.22) \quad K^{-1}((x_1, 2n), (2n, y_2)) = \frac{\mathbf{i}^{(4n-x_1-y_2+2)/2} q^{(4+4n-x_1-y_2)(x_1+y_2-4n)/4}}{(2\pi\mathbf{i})^2} \int_{\Gamma_{a,q}^n} \int_{\Gamma_0} dz dw$$

$$\frac{w^n}{z^{(x_1+1)/2}(w-z)} \frac{\prod_{k=0}^{n-1} (z + aq^{-2k+y_2-2})}{\prod_{k=0}^{(y_2-1)/2} (w + aq^{2k-1}) \prod_{k=0}^{n-(y_1+1)/2} (awq^{2k+2} - 1)}.$$

From the above expression, we first move the white vertices to $x \in \mathbb{W}$ by using the recurrence generated from $K.K^{-1} = \mathbb{I}$ when no white and black vertices are adjacent under the recurrence. That is, we use the equation

$$(4.23) \quad K^{-1}(x, y) = -K^{-1}(x + 2e_1, y)\delta_{x_1 < 2n-1} - a\mathbf{i}q^{x_1-2n}K^{-1}(x + e_1 + e_2, y)$$

$$- a\mathbf{i}q^{2n-x_1-1}K^{-1}(x + e_1 - e_2, y)\delta_{x_1 < 2n-1}$$

for $x = (x_1, x_2) \in \mathbb{W}$ with $x_1 > 0$ and $y \in \mathbb{B}$. We can then move the black vertices to $y \in \mathbb{B}$ using the recurrence $K^{-1}.K = \mathbb{I}$ provided no white and black vertices are adjacent under the recurrence. That is, we can use the equation

$$(4.24) \quad K^{-1}(x, y) = -K^{-1}(x, y + 2e_1)\delta_{y_2 < 2n-1} - a\mathbf{i}q^{2n-y_1}K^{-1}(x, y + e_1 + e_2)\delta_{y_2 < 2n-1}$$

$$- a\mathbf{i}q^{y_1-2n+1}K^{-1}(x, y + e_1 - e_2)$$

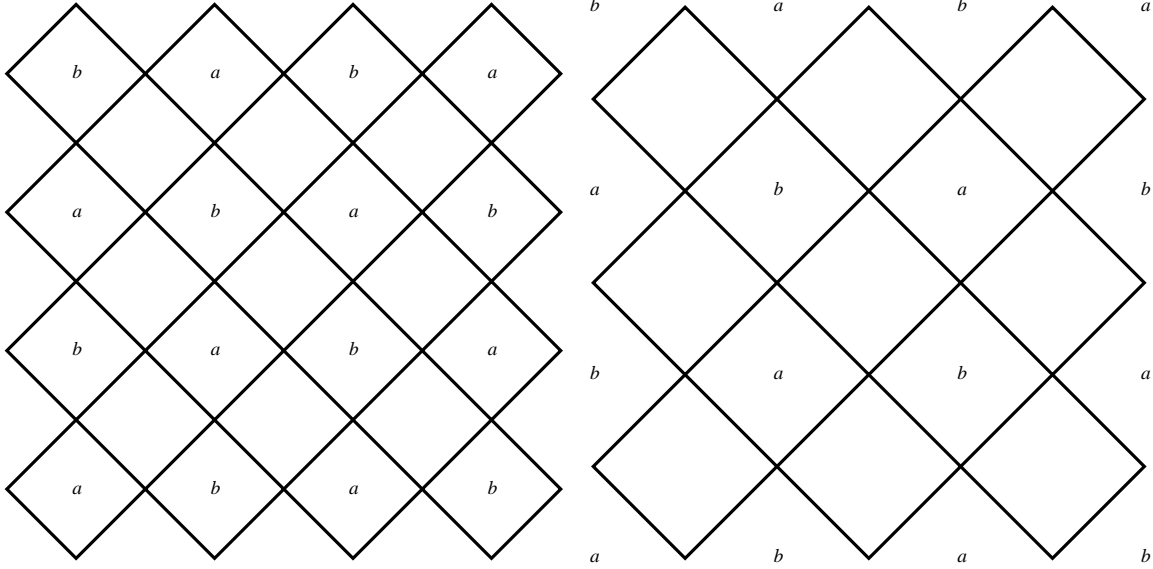
for $y = (y_1, y_2) \in \mathbb{B}$ with $y_2 > 0$ and $x \in \mathbb{W}$. These two recurrences give $f_1(x, y)$ defined in (4.3). If the vertices cross under the recurrence, then there is an extra factor counted from the contribution of $z = w$ in the expression of $f_1(x, y)$. This is exactly the expression $f_2(x, y)$. \square

5. TWO-PERIODIC WEIGHTING

In this section, we compute the generating function of the inverse Kasteleyn matrix for the two-periodic weighting of an Aztec diamond of size $4m$. This is much akin to the parameter a introduced in Section 3, except now there are two parameters, a and b , which decorate the edges of the aztec diamond in a checkerboard fashion described below. As remarked in the introduction, one of the special cases of this model is equivalent to a different, uniform tiling problem: namely, the so-called diabolos tilings of the fortress introduced in [27]. The dual graph of the fortress is the *square-octagon lattice* with certain boundary conditions. This model was the main motivation for this work.

Because of this periodicity, the problem becomes complicated in two ways: the recurrence relation increases in *order*, and the generating function becomes a *vector* of generating

FIGURE 9. The two-periodic weighting of the Aztec diamond with the picture on the left being an even ordered Aztec diamond and the picture on the right being an odd ordered Aztec diamond. The edges surrounding each face are given the same weight, denoted by the parameter in the center of each face with the exception of the boundary of the odd ordered Aztec diamond, where the edge weights are given by the adjacent parameter.



functions. Finally, the relation which we solve in order to compute the boundary generating function involves a certain matrix multiplication which must be explicitly diagonalized before the recurrence can be solved.

As the expressions are somewhat complicated, we can only write the overall generating function as linear combinations of the entries of the $K^{-1}(\mathbf{w}, \mathbf{b})$ where \mathbf{w} and \mathbf{b} are white and black vertices on the boundary of the Aztec diamond where the matrix K is given below. These entries of the inverse Kasteleyn matrix are computed explicitly and, for the bottom right hand boundary, are given in Lemma 5.1.

We consider a two-periodic weighting of the Aztec diamond and compute its boundary generating function. Due to the periodicity of the weights, we have two types of white and black vertices. We denote for $i \in \{0, 1\}$

$$(5.1) \quad \mathbf{W}_i = \{(x_1, x_2) : (x_1 + x_2) \bmod 4 = 2i + 1, (x_1, x_2) \in \mathbf{W}\}$$

and

$$(5.2) \quad \mathbf{B}_i = \{(x_1, x_2) : (x_1 + x_2) \bmod 4 = 2i + 1, (x_1, x_2) \in \mathbf{B}\}.$$

For an Aztec diamond of size n , we give weights a and b to the Aztec diamond in the following way: if the size of the Aztec diamond is even, i.e. $n = 2r$, then the edge weights around each face $(2i + 1, 2j + 1)$ for $0 \leq i, j \leq n - 1$ are given weight a if $(i + j) \bmod 2 = 0$ and weight b if $(i + j) \bmod 2 = 1$. Conversely, if $n = 2r - 1$, then the edge weights are obtained from embedding the diamond in an Aztec diamond of order $2r$. Figure 9 shows this choice of edge weights.

Let K denote the Kasteleyn matrix for an Aztec diamond when n is even and \tilde{K} denote the Aztec diamond when n is odd. We have

$$(5.3) \quad K(x, y) = \begin{cases} a(1-i) + bi & \text{if } y = x + e_1, x \in \mathbf{B}_i \\ (ai + b(1-i))i & \text{if } y = x + e_2, x \in \mathbf{B}_i \\ ai + b(1-i) & \text{if } y = x - e_1, x \in \mathbf{B}_i \\ (a(1-i) + bi)i & \text{if } y = x - e_2, x \in \mathbf{B}_i \end{cases}$$

and

$$(5.4) \quad \tilde{K}(x, y) = \begin{cases} ai + b(1-i) & \text{if } y = x + e_1, x \in \mathbf{B}_i \\ (ai + b(1-i))i & \text{if } y = x + e_2, x \in \mathbf{B}_i \\ a(1-i) + bi & \text{if } y = x - e_1, x \in \mathbf{B}_i \\ (a(1-i) + bi)i & \text{if } y = x - e_2, x \in \mathbf{B}_i \end{cases}$$

for $x \in \mathbf{B}$ and $y \in \mathbf{W}$.

We now give the entries of K^{-1} for white and black vertices on the bottom and left boundaries respectively for K is defined in (5.3) and the size of the Aztec diamond is even. To shorten the length of the formulas, we will write $[i]_2 = i \bmod 2$.

Lemma 5.1. *For an Aztec diamond of size n , let $n = 4m$ and let K denote the Kasteleyn matrix given in (5.3). Let $L(a, b, i, j)$ denote $K^{-1}((2i+1, 0), (0, 2j+1))$ for the Kasteleyn matrix with parameters a and b . We have*

$$(5.5) \quad L(a, b, i, j) = \frac{-\mathbf{i}^{i+j+1}}{(2\pi\mathbf{i})^2} \int_{|z|=1} \int_{|w|=1} \sum_{r=0}^{m-1} \sum_{k,l \in \{0,1\}} \frac{g_{2[i]_2+[j]_2+1}^{2k+l+1}(a, b, w, z) \alpha_k^r(a, b, w) \alpha_l^r(a, b, z)}{w^{[i/2]+1} z^{[j/2]+1}} dw dz$$

where for $1 \leq i, j \leq 4$ we have $g_i^j(a, b, w, z) = \mathbf{N}_{i,j}(a, b, w, z)$ which is given in Appendix A and

$$(5.6) \quad \alpha_k^r(a, b, w) = \frac{(\beta_0(a, b, w))^{2r} + (-1)^k (\beta_1(a, b, w))^{2r}}{4\sqrt{ab}(a^2 + b^2)(\sqrt{w}\sqrt{(b^4 + a^4)w + a^2b^2(1+w^2)})^k}$$

for $k \in \{0, 1\}$ with

$$(5.7) \quad \beta_l(a, b, w) = \frac{\left((a^2 + b^2)\sqrt{w} - (-1)^l \sqrt{(b^4 + a^4)w + a^2b^2(1+w^2)}\right)}{\sqrt{2ab(a^2 + b^2)}}$$

Note that the expressions $\alpha_0(a, b, w)$ and $\alpha_1(a, b, w)$ are polynomials in w and the $L(a, b, i, j)$ is a rational function in a and b .

The boundary inverse Kasteleyn entries have periodicity four with the size of the Aztec diamond. It is also possible to find the entries of the boundary inverse Kasteleyn matrix for the size of the Aztec diamond equal to $4m-1$, $4m-2$ and $4m-3$ but we will not state these results here. The proof of the lemma is given in the next section. For the generating function of K^{-1} , we need the following terms:

$$(5.8) \quad \begin{aligned} c_{GF}(w_1, w_2) &= 2(1 + a^2) + a(w_1^2 + w_1^{-2})(w_2^2 + w_2^{-2}), \\ s_{i,0}(w_1, w_2) &= -a(w_1^{-2}w_2^{-2} + w_1^2w_2^{-2}) - aiw_1^2 + aiw_1^{-2} - 2a^{2i}, \\ s_{i,2n}(w_1, w_2) &= w_2^{2n} \left(-a(w_1^2w_2^2 + w_1^{-2}w_2^2) + aiw_1^2 - aiw_1^{-2} - 2a^{2(1-i)} \right). \end{aligned}$$

for $i \in \{0, 1\}$. We set $f_n(x) = (1 - x^n)/(1 - x)$ — the sum of a geometric series and we also let for $n = 2r$ and $\mathbf{w} = (w_1, w_2)$ and $\mathbf{b} = (b_1, b_2)$,

$$(5.9) \quad \begin{aligned} d(\mathbf{w}, \mathbf{b}) &= f_r(w_2^4 b_2^4) f_r(w_1^4 b_1^4) b_2 w_1 (a + w_1^2 b_1^2 + w_2^2 b_2^2 (1 + a w_1^2 b_1^2)) (w_2^2 + b_1^2 - \mathbf{i}(1 + w_2^2 b_1^2)) \\ &+ ((1 - \mathbf{i} w_2^2) w_1^{-1} b_2 (1 + a w_2^2 b_2^2) + (w_2^2 - \mathbf{i}) w_1^{2n+1} b_1^{2n} b_2 (a + w_2^2 b_2^2)) f_r(w_2^4 b_2^4) \\ &- \frac{d_{\text{sides}}(\mathbf{w}, \mathbf{b})}{c_{GF}(b_1, b_2)} f_r(w_1^4 b_1^4) \end{aligned}$$

where

$$(5.10) \quad \begin{aligned} d_{\text{sides}}(\mathbf{w}, \mathbf{b}) &= s_{0,0}(w_1, w_2) (a(b_1^2 - \mathbf{i}) w_1 b_2 + w_1 b_2^{-1} (1 - \mathbf{i} b_1^2)) \\ &+ s_{1,0}(w_1, w_2) ((b_1^2 - \mathbf{i}) w_1^3 b_1^2 b_2 + a w_1^3 b_1^2 b_2^{-1} (1 - \mathbf{i} b_1^2)) \\ &+ s_{0,2n}(w_1, w_2) ((1 - \mathbf{i} b_1^2) w_1 b_2^{2n-1} + a w_1 b_2^{2n+1} (1 - \mathbf{i} b_1^2)) \\ &+ s_{1,2n}(w_1, w_2) (a(1 - \mathbf{i} b_1^2) w_1^3 b_1^2 b_2^{2n-1} + w_1^3 b_1^2 b_2^{2n+1} (1 - \mathbf{i} b_1^2)). \end{aligned}$$

We denote the generating function of K^{-1} for K defined in (5.3) as

$$(5.11) \quad G(a, b, \mathbf{w}, \mathbf{b}) = \sum_{x \in \mathbb{W}} \sum_{y \in \mathbb{B}} K^{-1}(x, y) \mathbf{w}^x \mathbf{b}^y$$

where $\mathbf{w}^x = w_1^{x_1} w_2^{x_2}$ and $\mathbf{b}^y = b_1^{y_1} b_2^{y_2}$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Theorem 5.2. *For an Aztec diamond of size n , the generating function of K^{-1} for K defined in (5.3) with $b = 1$, $n = 4m$ for $m \in \mathbb{N}$, $\mathbf{w} = (w_1, w_2)$ and $\mathbf{b} = (b_1, b_2)$ is given by*

$$(5.12) \quad \begin{aligned} G(a, 1, \mathbf{w}, \mathbf{b}) &= \frac{d(\mathbf{w}, \mathbf{b})}{c_{GF}(w_1, w_2)} \\ &+ \left(\sum_{i,j \in \{0,1\}} \sum_{k,l \in \{0,2n\}} \sum_{(x_1,k) \in \mathbb{W}_i} \sum_{(l,y_2) \in \mathbb{B}_j} \frac{s_{i,k}(w_1, w_2) s_{j,l}(b_2, b_1) K^{-1}((x_1, k), (l, y_2))}{c_{GF}(w_1, w_2) c_{GF}(b_1, b_2)} w_1^{x_1} b_2^{y_2} \right) \end{aligned}$$

where

$$(5.13) \quad \begin{aligned} K^{-1}((x_1, 0), (0, y_2)) &= L\left(a, 1, \frac{x_1 - 1}{2}, \frac{y_2 - 1}{2}\right), \\ K^{-1}((x_1, 0), (2n, y_2)) &= \mathbf{i}^{2n-1-x_1+y_2} L\left(1, a, n - \frac{x_1 + 1}{2}, \frac{y_2 - 1}{2}\right), \\ K^{-1}((x_1, 2n), (0, y_2)) &= \mathbf{i}^{2n-1+x_1-y_2} L\left(1, a, \frac{x_1 - 1}{2}, n - \frac{y_2 + 1}{2}\right) \text{ and} \\ K^{-1}((x_1, 2n), (2n, y_2)) &= L\left(a, 1, n - \frac{x_1 + 1}{2}, n - \frac{y_2 + 1}{2}\right). \end{aligned}$$

and $L(a, b, i, j)$ is given in Lemma 5.1.

For the proof of this theorem, we are not able to follow the approach exactly as given in Theorem 3.1 because the relations $K.K^{-1} = \mathbb{I}$ and $K^{-1}.K = \mathbb{I}$, while technically sufficient, are not of a suitably nice form to allow any progress. Instead, we need to use two further recurrences $K^*.K.K^{-1} = K^*\mathbb{I}$ and $K^{-1}.K.K^* = \mathbb{I}.K^*$ where K^* denotes complex conjugate transpose. These identities have an interpretation in terms of the discrete Laplacian interpretation, see [20] — though this interpretation is only heuristically relevant here.

5.1. Boundary Generating Function. In this section, we find the boundary recurrence relation and solve the recurrence. The method is similar to the one used in Section 3 but due to the two-periodic weighting, we have a different recurrence for each type of white and black vertex. This means we have a matrix equation explaining the recurrence which is also periodic. Due to the nature of the computations, we had to rely heavily on computer algebra in this subsection.

Let $Z_P(a, b, n)$ denote the partition function of an Aztec diamond of size $2n$ with edge weights a and b described above and let $\tilde{Z}_P(a, b, n)$ denote the partition function of an Aztec diamond of size $2n - 1$ with edge weights a and b as described above for an Aztec diamond of size $2n - 1$. Let $Z_{kl}(i, j, a, b, n)$ count the number of weighted tilings of an Aztec diamond of size $2n$ with vertices $(4i + 2k + 1, 0)$ and $(0, 4j + 2l + 1)$ removed with edge weights a and b . Let $\tilde{Z}_{kl}(i, j, a, b, n)$ count the number of weighted tilings of an Aztec diamond of size $2n - 1$ with vertices $(4i + 2k + 1, 0)$ and $(0, 4j + 2l + 1)$ removed with weights a and b .

We introduce the following generating functions: denote

$$(5.14) \quad G_{kl}(a, b, x, y, z) = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{Z_{kl}(i, j, a, b, n)}{Z_P(a, b, n)} x^i y^j z^n$$

and

$$(5.15) \quad \tilde{G}_{kl}(a, b, x, y, z) = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{\tilde{Z}_{kl}(i, j, a, b, n)}{\tilde{Z}_P(a, b, n)} x^i y^j z^n.$$

for $k, l \in \{0, 1\}$. Let

$$(5.16) \quad \mathbf{G}(a, b, x, y, z) = \begin{pmatrix} G_{00}(a, b, x, y, z) \\ G_{01}(a, b, x, y, z) \\ G_{10}(a, b, x, y, z) \\ G_{11}(a, b, x, y, z) \end{pmatrix}$$

and similarly, let $\tilde{\mathbf{G}}$ denote the corresponding vector for \tilde{G}_{kl} .

Lemma 5.3. *The boundary generating functions G and \tilde{G} satisfy the following recurrences*

$$(5.17) \quad \mathbf{G}(a, b, x, y, z) = \mathbf{A}(a, b, x, y) \cdot \tilde{\mathbf{G}}\left(\frac{1}{2a}, \frac{1}{2b}, x, y, z\right) + \mathbf{B}(a) \frac{z}{1-z}$$

and

$$(5.18) \quad \tilde{\mathbf{G}}(c, d, x, y, z) = z \mathbf{C}(c, d, x, y) \cdot \mathbf{G}(c, d, x, y, z) + \mathbf{D}(c, d) \frac{z}{1-z}$$

where

$$(5.19) \quad \mathbf{A}(a, b, x, y) = \begin{pmatrix} \frac{1}{4a^2} & \frac{y}{4a^2} & \frac{x}{4a^2} & \frac{xy}{4a^2} \\ \frac{1}{4ab} & \frac{y}{4ab} & \frac{x}{4ab} & \frac{xy}{4ab} \\ \frac{1}{4ab} & \frac{y}{4ab} & \frac{x}{4ab} & \frac{xy}{4ab} \\ \frac{1}{4b^2} & \frac{y}{4b^2} & \frac{x}{4b^2} & \frac{xy}{4b^2} \end{pmatrix}, \quad \mathbf{B}(a) = \begin{pmatrix} \frac{1}{2a} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(5.20) \quad \mathbf{C}(c, d, x, y) = \frac{1}{c^2 + d^2} \begin{pmatrix} d^2 & cdy & cdx & c^2xy \\ d^2 & cd & cdx & c^2x \\ d^2 & cdy & cd & c^2y \\ d^2 & cd & cd & c^2 \end{pmatrix} \quad \text{and} \quad \mathbf{D}(c, d) = \begin{pmatrix} \frac{c}{c^2 + d^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Proof. By urban renewal, edge contraction and removal of pendant edges, we can find that $Z_P(a, b, n) = (4ab)^{4n^2} \tilde{Z}_P(1/(2a), 1/(2b), n)$. We now write a relation for $Z_{st}(i, j, a, b, n)$ for $s, t \in \{0, 1\}$ following the same method given in Lemma 3.2. That is, we add two auxiliary vertices to the graph v_1 and v_2 and their two auxiliary edges which are edges from v_1 to the vertex at $(4i + 2s + 1, 0)$ and from v_2 to $(0, 4j + 2t + 1)$ so that the resulting graph is planar. We can then apply urban renewal, edge contraction, removal of pendant edges. When splitting the additional square from the auxiliary edge obtained from v_1 , we have an extra factor of $1/(2a^{1-s}b^s)$ for $s \in \{0, 1\}$ obtained from the edge weights of the additional square. We have a similar factor for the splitting the additional square from the auxiliary edge corresponding to v_2 . We find that for $s, t \in \{0, 1\}$

$$(5.21) \quad Z_{st}(i, j, a, b, n) = \sum_{k, l \in \{0, 1\}} \frac{(4ab)^{4n^2}}{4a^{2-s-t}b^{s+t}} \tilde{Z}_{kl} \left(i - k(1-s), j - l(1-t), \frac{1}{2a}, \frac{1}{2b}, n \right).$$

The boundary conditions for \tilde{Z}_{kl} are similar to those given in Lemma 3.2. We find that

$$(5.22) \quad \begin{aligned} Z_{st}(i, j, a, b, n) = & \sum_{\substack{k, l \in \{0, 1\} \\ (i-k, j-l) \neq (-1, -1)}} \frac{(4ab)^{4n^2}}{4a^{2-s-t}b^{s+t}} \tilde{Z}_{kl} \left(i - k(1-s), j - l(1-t), \frac{1}{2a}, \frac{1}{2b}, n \right) \\ & + \frac{(4ab)^{4n^2}}{2a} \tilde{Z}_P \left(\frac{1}{2a}, \frac{1}{2b}, n \right) \mathbb{I}_{(i, j, s, t) = (0, 0, 0, 0)} \end{aligned}$$

We can divide both sides of the above equation by $Z_P(a, b, n)$ which gives

$$(5.23) \quad \frac{Z_{st}(i, j, a, b, n)}{Z_P(a, b, n)} = \sum_{\substack{k, l \in \{0, 1\} \\ (i-k, j-l) \neq (-1, -1)}} \frac{\tilde{Z}_{kl} \left(i - k(1-s), j - l(1-t), \frac{1}{2a}, \frac{1}{2b}, n \right)}{4a^{2-s-t}b^{s+t} \tilde{Z}_P \left(\frac{1}{2a}, \frac{1}{2b}, n \right)} + \frac{\mathbb{I}_{(i, j, s, t) = (0, 0, 0, 0)}}{2a}$$

For the recurrence equations given in the equation above, we can multiply by $x^i y^j z^n$ and sum over the relevant quantities. This gives a system of equations involving the generating functions whose matrix equation is given by (5.17).

Analogously, we can find a recurrence relation for $\tilde{Z}_{st}(i, j, c, d, n)$. We find that for $s, t \in \{0, 1\}$,

$$(5.24) \quad \begin{aligned} \frac{\tilde{Z}_{st}(i, j, c, d, n)}{\tilde{Z}_P(a, b, n)} = & \sum_{\substack{k, l \in \{0, 1\} \\ (i-k, j-l) \neq (-1, -1)}} \frac{d^{2-k-l} c^{k+l}}{c^2 + d^2} \frac{Z_{kl}(i - k(1-s), j - l(1-t), c, d, n-1)}{Z_P(c, d, n-1)} \\ & + \frac{c}{c^2 + d^2} \mathbb{I}_{(i, j, s, t) = (0, 0, 0, 0)}. \end{aligned}$$

The above recurrence relation gives the generating function given in (5.18). \square

Let

$$(5.25) \quad \mathbf{M}(a, b, x, y) = \mathbf{A}(a, b, x, y) \cdot \mathbf{C}((2a)^{-1}, (2b)^{-1}, x, y) \cdot \mathbf{A}((2a)^{-1}, (2b)^{-1}, x, y) \cdot \mathbf{C}(a, b, x, y)$$

and define the vectors

$$(5.26) \quad \mathbf{B}_1(a, b, x, y) = \mathbf{A}(a, b, x, y) \cdot \mathbf{D}((2a)^{-1}, (2b)^{-1}) + \mathbf{B}(a)$$

and

$$(5.27) \quad \mathbf{B}_2(a, b, x, y) = \mathbf{A}(a, b, x, y) \cdot \mathbf{C}((2a)^{-1}, (2b)^{-1}, x, y) \cdot \mathbf{B}_1((2a)^{-1}, (2b)^{-1}, x, y).$$

Lemma 5.4. *For $n = 4m$, the n^{th} coefficient of z of the generating function $\mathbf{G}(a, b, x, y, z)$ is given by*

$$(5.28) \quad \sum_{i=0}^{m-1} \mathbf{\Gamma} \mathbf{\Lambda}^i \mathbf{\Gamma}^{-1} \cdot (\mathbf{B}_1 + \mathbf{B}_2)$$

where $(\mathbf{\Lambda}, \mathbf{\Gamma})$ is the eigensystem of $\mathbf{M}(a, b, x, y)$. Explicitly, the eigenvalues of $\mathbf{M}(a, b, x, y)$ are given by

$$(5.29) \quad \lambda_{2i+j+1}(a, b, x, y) = \beta_i(a, b, x)^2 \beta_j(a, b, y)^2$$

and the eigenvectors are given by

$$(5.30) \quad v_{2i+j+1} = \begin{pmatrix} \frac{((b^2 - a^2)x - (-1)^i \sqrt{x} \sqrt{(a^4 + b^4)x + a^2 b^2(1+x^2)})((b^2 - a^2)y - (-1)^j \sqrt{y} \sqrt{(a^4 + b^4)y + a^2 b^2(1+y^2)})}{a^2 b^2(1+x)(1+y)} \\ \frac{((b^2 - a^2)x - (-1)^i \sqrt{x} \sqrt{(a^4 + b^4)x + a^2 b^2(1+x^2)})}{ab(1+x)} \\ \frac{(b^2 - a^2)y - (-1)^j \sqrt{y} \sqrt{(a^4 + b^4)y + a^2 b^2(1+y^2)}}{ab(1+y)} \\ 1 \end{pmatrix}$$

Although the above lemma does give the boundary generating function for a two-periodic Aztec diamond, the expression is complicated. We believe that the expression given in Lemma 5.1 is more feasible for potential asymptotic computations.

Proof. From Lemma 5.3, we have a generating function equation for \mathbf{G} and $\tilde{\mathbf{G}}$. We can write

$$(5.31) \quad \begin{aligned} \mathbf{G}(a, b, x, y, z) &= z \mathbf{A}(a, b, x, y) \cdot \mathbf{C}((2a)^{-1}, (2b)^{-1}, x, y) \cdot \mathbf{G}((2a)^{-1}, (2b)^{-1}, x, y, z) \\ &+ (\mathbf{A}(a, b, x, y) \cdot \mathbf{D}((2a)^{-1}, (2b)^{-1}) + \mathbf{B}(a)) \frac{z}{1-z} \end{aligned}$$

Apply the above equation to itself, we can write

$$(5.32) \quad \begin{aligned} \mathbf{G}(a, b, x, y, z) &= z^2 \mathbf{M}(a, b, x, y) \mathbf{G}(a, b, x, y, z) + \frac{z}{1-z} \mathbf{B}_1(a, b, x, y) + \frac{z^2}{1-z} \mathbf{B}_2(a, b, x, y) \\ &= \sum_{k=0}^{\infty} z^{2k} (\mathbf{M}(a, b, x, y))^k \cdot \left(\frac{z}{1-z} \mathbf{B}_1(a, b, x, y) + \frac{z^2}{1-z} \mathbf{B}_2(a, b, x, y) \right) \end{aligned}$$

by using the expansion of a geometric series of matrices. The above equation can be solved but it does not appear to give a tractable answer. As we are interested in the n^{th} coefficient, we can use the above expansion and the expansion of $(1-z)^{-1}$ to find the coefficient of z^n . This is given by

$$(5.33) \quad \sum_{k=0}^{m-1} (\mathbf{M}(a, b, x, y))^k \cdot (\mathbf{B}_1(a, b, x, y) + \mathbf{B}_2(a, b, x, y)).$$

The eigenvalues and eigenvectors of \mathbf{M} can be verified by $\mathbf{M} \cdot v = \lambda v$ where λ is the eigenvalue for the eigenvector v . \square

We can now prove Lemma 5.1.

Proof of Lemma 5.1. From Lemma 5.4, we can write the n^{th} coefficient of $\mathbf{G}(a, b, x, y, z)$ as

$$(5.34) \quad \sum_{i=0}^{m-1} \sum_{j=1}^4 \mathbf{X}_j \lambda_j^i$$

where $\{\lambda_i\}_{i=1}^4$ are the eigenvalues of $\mathbf{M}(a, b, x, y)$ and \mathbf{X}_i are four column vectors which are the coefficients of a_i in the following expression

$$(5.35) \quad \mathbf{\Gamma} \cdot \text{diag}(a_1, a_2, a_3, a_4) \cdot \mathbf{\Gamma}^{-1} \cdot (\mathbf{B}_1(a, b, x, y) + \mathbf{B}_2(a, b, x, y)).$$

where $\text{diag}(a_1, a_2, a_3, a_4)$ denotes a diagonal matrix with four entries a_1, \dots, a_4 . We can rewrite (5.34) as

$$(5.36) \quad \sum_{k=0}^{m-1} \mathbf{Y} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^i \\ \lambda_2^i \\ \lambda_3^i \\ \lambda_4^i \end{pmatrix}$$

where

$$(5.37) \quad \mathbf{Y} = [\mathbf{X}_1 \quad \mathbf{X}_2 \quad \mathbf{X}_3 \quad \mathbf{X}_4] \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

A computation shows that for

$$(5.38) \quad \mathbf{D}_1 = 16ab(a^2 + b^2)^2 \text{diag} \begin{pmatrix} 1 \\ \frac{\sqrt{y}\sqrt{(b^4 + a^4)y + a^2b^2(1 + y^2)}}{\sqrt{x}\sqrt{(b^4 + a^4)x + a^2b^2(1 + x^2)}} \\ \frac{\sqrt{x}\sqrt{(b^4 + a^4)x + a^2b^2(1 + x^2)}}{\sqrt{y}\sqrt{(b^4 + a^4)y + a^2b^2(1 + y^2)}} \end{pmatrix}$$

that

$$(5.39) \quad \mathbf{N}(a, b, x, y) = \mathbf{Y} \cdot \mathbf{D}_1$$

where $\mathbf{N}(a, b, x, y)$ is the matrix defined in (A.1) and

$$(5.40) \quad \begin{pmatrix} \alpha_0^i(a, b, x) \alpha_0^i(a, b, y) \\ \alpha_0^i(a, b, x) \alpha_1^i(a, b, y) \\ \alpha_1^i(a, b, x) \alpha_0^i(a, b, y) \\ \alpha_1^i(a, b, x) \alpha_1^i(a, b, y) \end{pmatrix} = \mathbf{D}_1^{-1} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^i \\ \lambda_2^i \\ \lambda_3^i \\ \lambda_4^i \end{pmatrix}$$

where α_j is defined in the statement of Lemma 5.1 for $j \in \{0, 1\}$. This means that the coefficient of z^n in the expression $\mathbf{G}(a, b, x, y, z)$ is equal to

$$(5.41) \quad \mathbf{N}(a, b, x, y) \cdot \begin{pmatrix} \alpha_0^i(a, b, x) \alpha_0^i(a, b, y) \\ \alpha_0^i(a, b, x) \alpha_1^i(a, b, y) \\ \alpha_1^i(a, b, x) \alpha_0^i(a, b, y) \\ \alpha_1^i(a, b, x) \alpha_1^i(a, b, y) \end{pmatrix}.$$

We can extract the relevant coefficient of the above equation to find $|L(a, b, i, j)|$. As each matching of the Aztec diamond with the vertices $(2i + 1, 0)$ and $(0, 2j + 1)$ removed has the same sign, the sign of one such matching determines the sign of $L(a, b, i, j)$.

□

5.2. Generating Function of K^{-1} . We now prove Theorem 5.2.

For notational purposes in the proof, we write $G = G(a, 1, \mathbf{w}, \mathbf{b})$ and let

$$(5.42) \quad G|_{x=(i,j)} = \sum_{\substack{x_1=i, x_2=j \\ y \in \mathbf{B}}} K^{-1}(x, y) w^x b^y$$

for $i \in \{1, 2n-1\}$ or $j \in \{0, 2n\}$. For $j \in \{0, 2n\}$, we also write

$$(5.43) \quad G|_{x=(x_1,j)} = \sum_{\substack{1 \leq i \leq 2n-1, \\ i \bmod 2=1}} G|_{x=(i,j)}$$

and for $i \in \{1, 2n-1\}$

$$(5.44) \quad G|_{x=(i,x_2)} = \sum_{\substack{0 \leq j \leq 2n, \\ i \bmod 2=0}} G|_{x=(i,j)}.$$

Proof of Theorem 5.2. Recall that K^* is the conjugate transpose of K . Applying K^* to both sides of the equation $K.K^{-1} = \mathbb{I}$, we obtain

$$(5.45) \quad \begin{aligned} & a(K^{-1}(x + 2e_1, y)\delta_{x_2 < 2n}\delta_{x_1 < 2n-1} + K^{-1}(x + 2e_2, y)\delta_{x_2 < 2n}\delta_{x_1 > 1} \\ & + K^{-1}(x - 2e_1, y)\delta_{x_2 > 0}\delta_{x_1 > 1} + K^{-1}(x - 2e_2, y)\delta_{x_2 > 0}\delta_{x_1 < 2n-1}) \\ & + 2(1 + a^2 - \delta_{x_2=0}(\delta_{x \in \mathbf{W}_0} + a^2\delta_{x \in \mathbf{W}_1}) - \delta_{x_2=2n}(a^2\delta_{x \in \mathbf{W}_0} + \delta_{x \in \mathbf{W}_1})) K^{-1}(x, y) \\ & + a i \delta_{x_2=0} (-K^{-1}(x + e_2 - e_1, y)\delta_{x_1 > 1} + K^{-1}(x + e_1 - e_2, y)\delta_{x_1 < 2n-1}) \\ & + a i \delta_{x_2=2n} (K^{-1}(x + e_2 - e_1, y)\delta_{x_1 > 1} - K^{-1}(x + e_1 - e_2, y)\delta_{x_1 < 2n-1}) = K^*(\delta_{\tilde{x}=y}(x)) \end{aligned}$$

for $x = (x_1, x_2) \in \mathbf{W}$ and $y \in \mathbf{B}$ and where $K^*(\delta_{\tilde{x}=y}(x))$ means K^* is applied to the white vertex y when it equal to the vertex \tilde{x} . We also have a similar expression for a recurrence that moves the black vertices.

We can multiply both sides of equation (5.45) by $\mathbf{w}^x \mathbf{b}^y = w_1^{x_1} w_2^{x_2} b_1^{y_1} b_2^{y_2}$ and sum over all the white and black vertices of the Aztec diamond. We can then rewrite each term of (5.45) using $G(a, 1, \mathbf{w}, \mathbf{b})$, expression with white vertices on the boundary and expression with $x_1 = 1$ or $x_1 = 2n-1$. For example, we have

$$(5.46) \quad \begin{aligned} & \sum_{\substack{x \in \mathbf{W} \\ y \in \mathbf{B}}} \mathbf{w}^x \mathbf{b}^y K^{-1}(x + 2e_1, y)\delta_{x_2 < 2n}\delta_{x_1 < 2n-1} \\ & = \frac{G(a, 1, \mathbf{w}, \mathbf{b}) - \left(\sum_{\substack{x_1=1, x_2 \neq 0 \\ x \in \mathbf{W}, y \in \mathbf{B}}} + \sum_{\substack{x_2=0 \\ x \in \mathbf{W}, y \in \mathbf{B}}} \right) \mathbf{w}^x \mathbf{b}^y K^{-1}(x, y)}{w_1^2 w_2^2}. \end{aligned}$$

We can simplify the rest of the terms of (5.45) and combine them. We obtain

$$\begin{aligned}
(5.47) \quad & (2(1+a^2) + (w_1^2 + w_1^{-2})(w_2^2 + w_2^{-2}))G \\
& - a G|_{x=(1,x_2)} \left(\frac{1+w_2^4}{w_1^2 w_2^2} \right) - a G|_{x=(2n-1,x_2)} \left(\frac{w_1^2(1+w_2^4)}{w_2^2} \right) \\
& + a G|_{x=(1,0)} \left(\frac{1+w_2^4}{w_1^2 w_2^2} - \frac{w_2^2}{w_1^2} - w_1^{-2} \mathbf{i} \right) + a G|_{x=(1,2n)} \left(\frac{1+w_2^4}{w_1^2 w_2^2} - \frac{1}{w_1^2 w_2^2} + w_1^{-2} \mathbf{i} \right) \\
& + a G|_{x=(2n-1,0)} \left(\frac{w_1^2(1+w_2^4)}{w_2^2} - w_1^2 w_2^2 + w_1^2 \mathbf{i} \right) + a G|_{x=(2n-1,2n)} \left(\frac{w_1^2(1+w_2^4)}{w_2^2} - \frac{w_1^2}{w_2^2} - w_1^2 \mathbf{i} \right) \\
& + G|_{x=(x_1,0)} \left(-a(w_1^{-2} w_2^{-2} + w_1^2 w_2^{-2}) - a w_1^2 \mathbf{i} + a w_1^{-2} \mathbf{i} - 2 \right) \\
& + G|_{x=(x_1,0)} \left(-a(w_1^{-2} w_2^{-2} + w_1^2 w_2^{-2}) - a w_1^2 \mathbf{i} + a w_1^{-2} \mathbf{i} - 2a^2 \right) \\
& + G|_{x=(x_1,2n)} \left(-a(w_1^2 w_2^2 + w_1^{-2} w_2^2) + a w_1^2 \mathbf{i} - a w_1^{-2} \mathbf{i} - 2a^2 \right) \\
& + G|_{x=(x_1,2n)} \left(-a(w_1^2 w_2^2 + w_1^{-2} w_2^2) + a w_1^2 \mathbf{i} - a w_1^{-2} \mathbf{i} - 2 \right) = \sum_{\substack{x \in \mathbb{W} \\ y \in \mathbb{B}}} K^*(\delta_{\bar{x}=y}(x))
\end{aligned}$$

The terms $G|_{x=(1,x_2)}$ and $G|_{x=(2n-1,x_2)}$ involve white vertices away from the boundary. We can use the recurrence relation from $K.K^{-1} = \mathbb{I}$ to write these expressions in terms of the boundary vertices. For $G|_{x=(1,x_2)}$, we use the recurrence relation

$$(5.48) \quad K^{-1}(x + e_2 + e_1, y) = \frac{1}{a^{1-i}} \delta_{x+e_2=y} - \mathbf{i} K^{-1}(x, y)$$

for $x = (1, x_2) \in \mathbb{W}_i$ for $i \in \{0, 1\}$. We can multiply the above equation by $w^x b^y$ and sum over the black and white vertices. We find that (5.48) becomes

$$(5.49) \quad w_2^{-2} G|_{x=(1,x_2)} - w_2^2 G|_{x=(1,0)} = \sum_{i \in \{0,1\}} \sum_{\substack{x_1=1 \\ x_2 \neq 2n, x \in \mathbb{W}_i \\ y \in \mathbb{B}}} \frac{1}{a^{1-i}} \delta_{x+e_2=y} w^x b^y - \mathbf{i} G|_{x=(1,0)} + \mathbf{i} G|_{x=(1,2n)}.$$

We can write

$$(5.50) \quad \sum_{i \in \{0,1\}} \sum_{\substack{x_1=1 \\ x_2 \neq 2n, x \in \mathbb{W}_i \\ y \in \mathbb{B}}} \frac{1}{a^{1-i}} \delta_{x+e_2=y} w^x b^y = f_{n/2}(w_2^4 b_2^4) w_1 b_2 (a^{-1} + w_2^2 b_2^2)$$

by evaluating each sum separately. Using (5.49) and the above equation, we find that

$$(5.51) \quad G|_{x=(1,x_2)} (w_2^{-2} + \mathbf{i}) = f_{n/2}(w_2^4 b_2^4) w_1 b_2 (a^{-1} + w_2^2 b_2^2) + w_2^{-2} G|_{x=(1,0)} + \mathbf{i} G|_{x=(1,2n)}.$$

We can do a similar procedure to find $G|_{x=(2n-1,x_2)}$ using the recurrence

$$(5.52) \quad K^{-1}(x + e_2 + e_1, y) \mathbf{i} = \frac{1}{a^i} \delta_{x+e_1=y} - K^{-1}(x, y)$$

for $x = (2n - 1, x_2) \in \mathbb{W}_i$ for $i \in \{0, 1\}$. By performing a similar procedure as above, we obtain

$$(5.53) \quad G|_{x=(2n-1, x_2)} (w_2^{-2} \mathbf{i} + 1) = f_{n/2}(w_2^4 b_2^4) w_1^{2n-1} b_1^{2n} b_2 (1 + w_2^2 b_2^2 a^{-1}) + w_2^{-2} \mathbf{i} G|_{x=(2n-1, 0)} + G|_{x=(2n-1, 2n)}$$

We can substitute equation (5.51) and (5.53) back into (5.47). We find that the coefficient of $a G|_{x=(1, 0)}$ in (5.47) is given by

$$(5.54) \quad \frac{1 + w_2^4}{w_1^2 w_2^2} - \frac{w_2^2}{w_1^2} - w_1^{-2} \mathbf{i} - \frac{w_2^2 (1 + w_2^4)}{(w_2^{-2} + \mathbf{i}) w_1^2 w_2^2} = 0.$$

Similarly, we also find that the coefficients of $a G|_{x=(1, 2n)}$, $a G|_{x=(2n-1, 0)}$ and $a G|_{x=(2n-1, 2n)}$ are zero. This means we can reduce (5.47) to

$$(5.55) \quad \begin{aligned} & c_{GF}(w_1, w_2) G - \frac{a(1 + w_2^4) w_1 b_2 (a^{-1} + w_2^2 b_2^2)}{w_1^2 w_2^2 (w_2^{-2} + \mathbf{i})} f_{n/2}(b_2^4 w_2^4) \\ & - \frac{a(1 + w_2^4) w_1^{2n+1} b_1^{2n} b_2 (1 + w_2^2 b_2^2 a^{-1})}{w_2^2 (w_2^{-2} \mathbf{i} + 1)} f_{n/2}(b_2^4 w_2^4) \\ & \sum_{i \in \{0, 1\}} s_{i, 0}(w_1, w_2) G|_{x=(x_1, 0)} + \sum_{i \in \{0, 1\}} s_{i, 2n}(w_1, w_2) G|_{x=(x_1, 2n)} = \sum_{\substack{x \in \mathbb{W} \\ y \in \mathbb{B}}} K^*(\delta_{\tilde{x}=y})(x). \end{aligned}$$

To evaluate $K^*(\delta_{\tilde{x}=y}(x))$ we have that

$$(5.56) \quad K^*(\delta_{\tilde{x}=y}(x)) = \delta_{x_2 > 0} (a^i \delta_{x-e_1=y} - \mathbf{i} a^i \delta_{x-e_2=y}) + \delta_{x_2 < 2n} (a^{1-i} \delta_{x+e_1=y} - \mathbf{i} a^{1-i} \delta_{x+e_2=y}).$$

We can multiply both sides by $\mathbf{w}^x \mathbf{b}^y$ and we obtain

$$(5.57) \quad \begin{aligned} & \sum_{\substack{x \in \mathbb{W} \\ y \in \mathbb{B}}} \mathbf{w}^x \mathbf{b}^y K^*(\delta_{\tilde{x}=y}(x)) = \\ & \sum_{i \in \{0, 1\}} \sum_{x \in \mathbb{W}_i} (w_1 b_1)^{x_1} (w_2 b_2)^{x_2} \left(-a^i \mathbf{i} \frac{b_1}{b_2} \delta_{x_2 > 0} + \frac{a^i}{b_1 b_2} \delta_{x_2 > 0} + a^{1-i} b_1 b_2 \delta_{x_2 < 2n} - a^{1-i} \mathbf{i} \frac{b_2}{b_1} \delta_{x_2 < 2n} \right). \end{aligned}$$

This can be evaluated explicitly and gives the first expression of $d(\mathbf{w}, \mathbf{b})$. Hence we obtain an expression for $G(a, 1, w, b)$ as linear combinations of $K^{-1}((x_1, 0), y)$ and $K^{-1}((x_1, 2n), y)$ for $1 \leq x_1 \leq 2n - 1$ and $y \in \mathbb{B}$.

Similar to (5.45), we can write a recurrence relation which moves the black vertices by using the recurrence $K^{-1}.K.K^* = K^*$. Using the analogous steps as above but for the black vertices, we reduce the generating function $\sum_{y \in \mathbb{B}} \mathbf{b}^y K^{-1}(x, y)$ as a linear combination of $K^{-1}(x, (0, y_2))$, $K^{-1}(x, (2n, y_2))$ and additive terms which are all dependent on the white vertices. These additive terms are given by d_{sides} . We can substitute back each expression that we obtain for $\sum_{y \in \mathbb{B}} \mathbf{b}^y K^{-1}(x, y)$ into (5.57). Evaluating gives (5.12) in terms of the $K^{-1}(x, y)$ where x is a white vertex and y is a black vertex, both on the boundary of the Aztec diamond.

We have computed $K^{-1}((x_1, 0), (0, y_2))$ in Lemma 5.1. By symmetry, we have that

$$K^{-1}((x_1, 0), (0, y_2)) = K^{-1}((2n - x_1, 2n), (2n, 2n - y_2))$$

which gives the last equation in (5.13). To compute $|K^{-1}((x_1, 2n), (0, 2n - y_2))|$, we can use the expression from $|K^{-1}((x_1, 0), (0, y_2))|$ provided we interchange between a and b .

The correct sign of $K^{-1}((x_1, 2n), (0, 2n - y_2))$ is given in the third equation of (5.13) and is determined by computing the sign of one perfect matching of the Aztec diamond when removing the vertices $(x_1, 2n)$ and $(0, 2n - y_2)$. From $K^{-1}((x_1, 2n), (0, 2n - y_2))$, we can find $K^{-1}((2n - x_1, 0), (2n, y_2))$ by symmetry. \square

APPENDIX A. THE MATRIX \mathbf{N}

In this subsection, we give the matrix \mathbf{N} . We have

$$\begin{aligned}
 \mathbf{N}(a, b, w, z)_{1,1} &= 4b^5(1 + wz) + a^2b^3(7 + 3w + 3z + 5wz) + 2a^4b(2 + w + z + wz) \\
 \mathbf{N}(a, b, w, z)_{1,2} &= -4b^7z(1 + wz) - a^2b^5z(5 + 5w + 5z + 5wz + 2wz^2) \\
 &\quad - a^4b^3z(2 + 6w + 5z + 5wz + z^2 + wz^2) - 2a^6bz(w + z + wz) \\
 \mathbf{N}(a, b, w, z)_{1,3} &= -4b^7w(1 + wz) - a^2b^5w(5 + 5w + 5z + 5wz + 2w^2z) \\
 &\quad - a^4b^3w(2 + 5w + w^2 + 6z + 5wz + w^2z) - 2a^6bw(w + z + wz) \\
 \mathbf{N}(a, b, w, z)_{1,3} &= 4b^9wz(1 + wz) + a^2b^7wz(3 + 7w + 7z + 5wz + 2w^2z + 2wz^2) \\
 &\quad + a^4b^5wz(3 + 9w + 2w^2 + 9z + 10wz + w^2z + 2z^2 + wz^2 + w^2z^2) \\
 &\quad + a^6b^3wz(7 + 6w + w^2 + 6z + 7wz + w^2z + z^2 + wz^2) \\
 &\quad + 2a^8bwz(2 + w + z + wz)
 \end{aligned}
 \tag{A.1}$$

$$\begin{aligned}
 \mathbf{N}(a, b, w, z)_{2,1} &= 2ab^4(1 + w + wz) + a^3b^2(5 + 7w + z + wz) + 4a^5(1 + w) \\
 \mathbf{N}(a, b, w, z)_{2,2} &= -2ab^6(2 + z + wz + wz^2) - a^3b^4(7 + 3w + 7z + 3wz + 2z^2 + 6wz^2) \\
 &\quad - a^5b^2(4 + 2w + 7z + 7wz + 3z^2 + 3wz^2) - 4a^7(1 + w)z \\
 \mathbf{N}(a, b, w, z)_{2,3} &= -2ab^6w(1 + w + wz) - a^3b^4w(6 + 9w + w^2 + 2z + wz + w^2z) \\
 &\quad - a^5b^2w(9 + 9w + 2w^2 + z + wz) - 4a^7w(1 + w) \\
 \mathbf{N}(a, b, w, z)_{2,4} &= 2ab^8w(2 + z + wz + wz^2) \\
 &\quad + a^3b^6w(5 + 5w + 6z + 3wz + w^2z + 3z^2 + 6wz^2 + w^2z^2) \\
 &\quad + a^5b^4w(2 + 5w + w^2 + 5z + 10wz + w^2z + 7z^2 + 5wz^2 + 2w^2z^2) \\
 &\quad + a^7b^2w(2w + 7z + 9wz + 2w^2z + 3z^2 + 3wz^2) + 4a^9w(1 + w)z
 \end{aligned}
 \tag{A.2}$$

(A.3)

$$\begin{aligned}
\mathbf{N}(a, b, w, z)_{3,1} &= 2ab^4(1 + z + wz) + a^3b^2(5 + w + 7z + wz) + 4a^5(1 + z) \\
\mathbf{N}(a, b, w, z)_{3,2} &= -2ab^6z(1 + z + wz) - a^3b^4z(6 + 2w + 9z + wz + z^2 + wz^2) \\
&\quad - a^5b^2z(9 + w + 9z + wz + 2z^2) - 4a^7z(1 + z) \\
\mathbf{N}(a, b, w, z)_{3,3} &= -2ab^6(2 + w + wz + w^2z) - a^3b^4(7 + 7w + 2w^2 + 3z + 3wz + 6w^2z) \\
&\quad - a^5b^2(4 + 7w + 3w^2 + 2z + 7wz + 3w^2z) - 4a^7w(1 + z) \\
\mathbf{N}(a, b, w, z)_{3,4} &= 2ab^8z(2 + w + wz + w^2z) \\
&\quad + a^3b^6z(5 + 6w + 3w^2 + 5z + 3wz + 6w^2z + wz^2 + w^2z^2) \\
&\quad + a^5b^4z(2 + 5w + 7w^2 + 5z + 10wz + 5w^2z + z^2 + wz^2 + 2w^2z^2) \\
&\quad + a^7b^2z(7w + 3w^2 + 2z + 9wz + 3w^2z + 2wz^2) + 4a^9wz(1 + z)
\end{aligned}$$

(A.4)

$$\begin{aligned}
\mathbf{N}(a, b, w, z)_{4,1} &= a^2b^3(3 + w + z + wz) + 2a^4b(5 + w + z) + 8a^6b^{-1} \\
\mathbf{N}(a, b, w, z)_{4,2} &= -a^2b^5(2 + z + wz + z^2 + wz^2) - a^4b^3(5 + w + 5z + wz + 6z^2 + 2wz^2) \\
&\quad - 2a^6b(2 + 5z + wz + 3z^2) - 8a^8b^{-1}z \\
\mathbf{N}(a, b, w, z)_{4,3} &= -a^2b^5(2 + w + w^2 + wz + w^2z) - a^4b^3(5 + 5w + 6w^2 + z + wz + 2w^2z) \\
&\quad - 2a^6b(2 + 5w + 3w^2 + wz) - 8a^8b^{-1}w \\
\mathbf{N}(a, b, w, z)_{4,4} &= a^2b^7(4 + 2w + 2z + 3wz + w^2z + wz^2 + w^2z^2) \\
&\quad + a^4b^5(7 + 7w + 2w^2 + 7z + 6wz + 3w^2z + 2z^2 + 3wz^2 + 5w^2z^2) \\
&\quad + a^6b^3(4 + 7w + 3w^2 + 7z + 9wz + 6w^2z + 3z^2 + 6wz^2 + 5w^2z^2) \\
&\quad + 2a^8b(2w + 2z + 5wz + 3w^2z + 3wz^2) + 8a^{10}b^{-1}wz
\end{aligned}$$

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